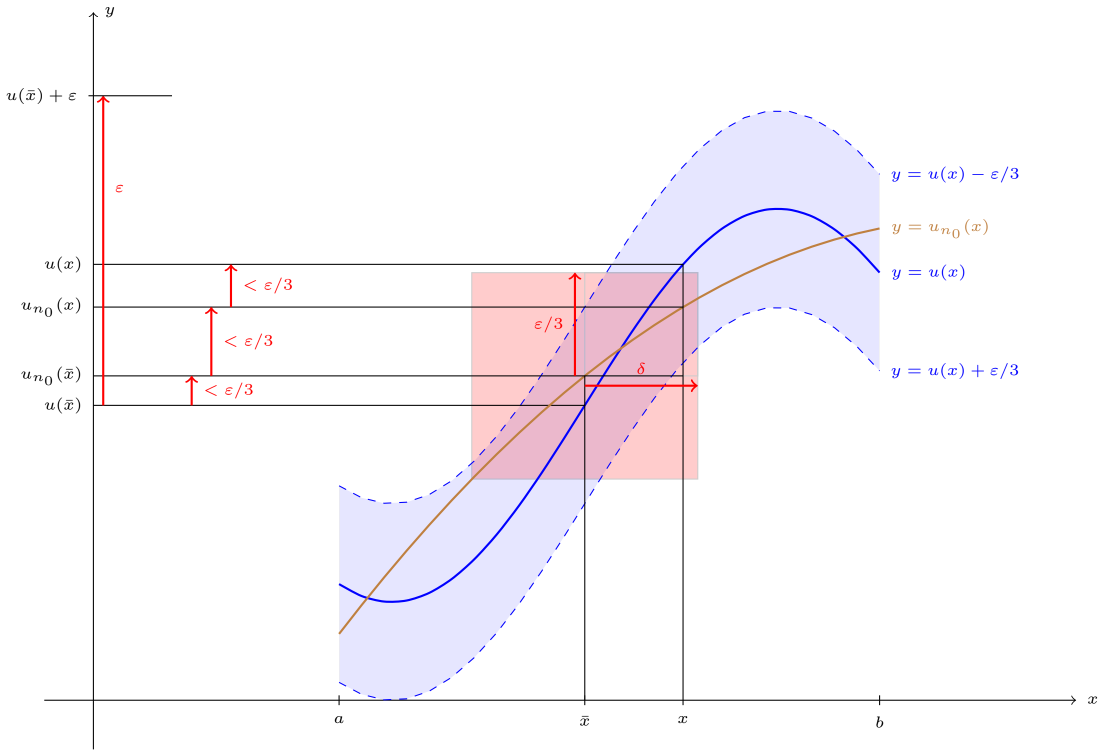
ECOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

LECTURE NOTES

Advanced Analysis

Academic year 2016–17



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The Lebesgue integral

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Measuring sets

Our aim

To extend:

- the notion of length of an interval in $\mathbb{R}$ to more complex subsets of $\mathbb{R}$;
- the notion of the area of a rectangle in $\mathbb{R}^2$ to more complex subsets of $\mathbb{R}^2$;
- the notion of volume of a cube in $\mathbb{R}^3$ to more complex subsets of $\mathbb{R}^3$;
- $\dots$.

Remark 1. *Such extensions cannot be constructed for all subsets of* $\mathbb{R}^p$ *for* $p \in \mathbb{N}^* = \{1, 2, 3, \dots\}$.

Some "super numbers"—our "contract"

We introduce "super numbers" $+\infty$ and $-\infty$ that must not be confused with the limits with the same notations.

The following rules will hold:

- $\forall a \in \mathbb{R}, \quad -\infty < a < +\infty$;
- $\forall a \in \mathbb{R}, \quad a + (\pm\infty) = (\pm\infty) + a = \pm\infty$;
- $\forall a \in \mathbb{R}$ with $a > 0, \quad a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty$;
- $\forall a \in \mathbb{R}$ with $a < 0, \quad a \cdot (\pm\infty) = (\pm\infty) \cdot a = \mp\infty$;
- $(+\infty) + (+\infty) = +\infty$ and $(-\infty) + (-\infty) = -\infty$;
- $0 \cdot (\pm\infty) = (\pm\infty) \cdot 0 = 0$.

Remark 2. $(+\infty) + (-\infty)$, $(-\infty) + (+\infty)$ *are* not *defined.*

Some notations

Let X be a "universe"; by this we mean a *non-empty* set.

- The family of all subsets of X is denoted by $\mathscr{P}(X)$:

  $$\mathscr{P}(X) := \{A \ : \ A \subset X\}.$$

  Remark that $\varnothing$, $X \in \mathscr{P}(X)$.

- We put

  $$\mathbb{R}_+ := \{a \in \mathbb{R} \ : \ a \geq 0\} = [0, +\infty[$$

  and

  $$\overline{\mathbb{R}}_+ := \{a \in \mathbb{R} \ : \ a \geq 0\} \cup \{+\infty\} = [0, +\infty].$$

1.1. Measurable sets and measures

The notion of σ-algebra

Definition 3.
Given:
- a universe X and
- a family of subsets $\mathscr{A} \subset \mathscr{P}(X)$

we say: $\mathscr{A}$ is a σ-algebra (on X) iff:

1. $\varnothing, X \in \mathscr{A}$;
2. $\forall A \in \mathscr{A}, \quad \complement A := X \setminus A \in \mathscr{A}$
3. for all sequences $\{A_n\}_{n=1}^{+\infty}$ of subsets in $\mathscr{A}$, we have $\cup_{n=1}^{+\infty} A_n \in \mathscr{A}$.

Remark 4. *$\mathscr{P}(X)$ is a σ-algebra on X, but for our purposes, this family is to large!*

The behaviour of σ-algebras under set-operations

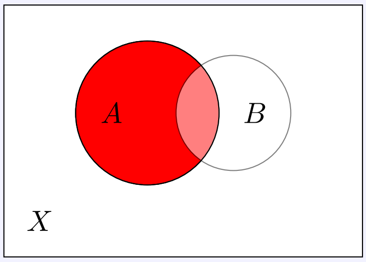
Proposition 5.

Hyp $\mathscr{A} \subset \mathscr{P}(X)$ *a σ-algebra*
Concl

1. *$\mathscr{A}$ is $\complement$-stable:*
$$\forall A \in \mathscr{A}, \quad \complement A \in \mathscr{A}.$$
2. *$\mathscr{A}$ is $\cup$-stable:*
$$\forall A, B \in \mathscr{A}, \quad A \cup B \in \mathscr{A}.$$
3. *$\mathscr{A}$ is $\cap$-stable:*
$$\forall A, B \in \mathscr{A}, \quad A \cap B = \complement\left(\complement A \cup \complement B\right) \in \mathscr{A}.$$

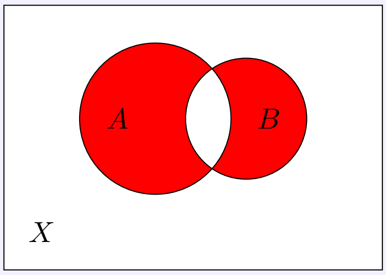
4. $\mathscr{A}$ *is* $\setminus$*-stable:*

$$\forall A, B \in \mathscr{A}, \quad A \setminus B = A \cap \complement B \in \mathscr{A}.$$



5. $\mathscr{A}$ *is* $\triangle$*-stable:*

$$\begin{aligned} \forall A, B \in \mathscr{A} \quad A \triangle B &= (A \setminus B) \cup (B \setminus A) \\ &= (A \cup B) \setminus (A \cap B) \in \mathscr{A} \end{aligned}$$



6. $\mathscr{A}$ *is* $\bigcup_{n=1}^{+\infty}$*-stable: For any* countable *index set* I *such as*

$$I = \mathbb{N} \quad \text{or} \quad I = \mathbb{Z} \quad \text{or} \quad I = \mathbb{Z}^* \quad \text{or} \quad \dots$$

we have

$$A_\iota \in \mathscr{A} \text{ for } \iota \in I \Longrightarrow \bigcup_{\iota \in I} A_\iota \in \mathscr{A},$$

where $\bigcup_{\iota \in I} A_\iota = \{x \in X \ : \ \exists \iota \in I \text{ such that } x \in A_\iota\}$.

7. $\mathscr{A}$ is $\bigcap_{n=1}^{+\infty}$-stable: For any countable *index set I such as*

$$I = \mathbb{N} \quad or \quad I = \mathbb{Z} \quad or \quad I = \mathbb{Z}^* \quad or \quad \ldots$$

we have

$$A_\iota \in \mathscr{A} \text{ for } \iota \in I \Longrightarrow \bigcap_{\iota \in I} A_\iota \in \mathscr{A},$$

where $\bigcap_{\iota \in I} A_\iota = \{x \in X \ : \ x \in A_\iota, \quad \forall \iota \in I\}$.

The notion of measurable space

Definition 6.
If $\mathscr{A}$ is a σ-algebra on the universe X, we say that the pair $(X, \mathscr{A})$ is a *measurable space*.

How to define σ-algebras

σ-algebras are most of the time huge families that cannot be defined by enumerating the subsets belonging to it. We will now introduce a way to define a σ-algebra that relies on the following result:

Proposition 7.
Any intersection

$$\bigcap_{\iota \in I} \mathscr{A}_\iota$$

of σ-algebras $\mathscr{A}_\iota$ ($\iota \in I$) on a common universe X is a σ-algebra on X,too.

Let $\mathscr{E} \subset \mathscr{P}(X)$ be a non-empty family of subset of a universe X.

We may then consider the intersection of all σ-algebras on X containing $\mathscr{E}$; we thus may consider

$$\sigma(\mathscr{E}) := \bigcap_{\substack{\mathscr{E} \subset \mathscr{A} \\ \mathscr{A} \text{ a } \sigma\text{-algebra}}} \mathscr{A}.$$

Thus, a subset A belongs to $\sigma(\mathscr{E})$ if and only if this subset A belongs to every σ-algebra containing $\mathscr{E}$.

Clearly, every subset $A \in \mathscr{E}$ belongs to $\sigma(\mathscr{E})$, too.

The notion of the σ-algebra generated by a family $\mathscr{E}$.

Proposition 8.
For any given, non-empty family $\mathscr{E}$ of subsets of a universe X,

$$\sigma(\mathscr{E}) := \bigcap_{\substack{\mathscr{E} \subset \mathscr{A} \\ \mathscr{A} \text{ a } \sigma\text{-algebra}}} \mathscr{A}$$

is the smallest σ-algebra on X containing $\mathscr{E}$.

Definition 9.

1. The generator of $\sigma(\mathscr{E})$:

   the non-empty family $\mathscr{E}$

2. The σ-algebra generated by $\mathscr{E}$:

   $$\sigma(\mathscr{E}) = \bigcap_{\substack{\mathscr{E} \subset \mathscr{A} \\ \mathscr{A} \text{ a } \sigma\text{-algebra}}} \mathscr{A},$$ i.e. the smallest σ-algebra on X containing $\mathscr{E}$.

The notion of a measure

Definition 10.
Given: a σ-algebra $\mathscr{A}$ on a universe X
we say: μ is a measure on $\mathscr{A}$ iff:
μ is a mapping

$$\mu : \mathscr{A} \to [0, +\infty], \quad A \mapsto \mu(A)$$

with the following properties

1. $\mu(\varnothing) = 0$;
2. For any sequence $\{A_n\}_{n=1}^{+\infty}$ of *pairwise disjoint* subsets in $\mathscr{A}$, we have
$$\mu\left(\dot{\bigcup_{n=1}^{+\infty}} A_n\right) = \sum_{n=1}^{+\infty} \mu(A_n),$$
i.e. μ is σ-additive.

Remark 11.

- *If one of the numbers* $\mu(A_n)$ *is equal to* $+\infty$*, the sum* $\sum_{n=1}^{+\infty} \mu(A_n)$ *is interpreted as being* $+\infty$.
- *If the sum* $\sum_{n=1}^{+\infty} \mu(A_n)$ *diverges, this sum is interpreted as being* $+\infty$.

Remark 12. *Every measure is* additive*, too, i.e.*

$$A, B \in \mathscr{A} \text{ with } A \cap B = \varnothing \Longrightarrow \mu(A \cup B) = \mu(A) + \mu(B)$$

(with $(+\infty) + a = +\infty$*, aso.) In order to see this, take* $A_1 := A$*,* $A_2 := B$*,* $A_n = \varnothing$ *for* $n > 2$ *in the definition of the* σ*-additivity!*

The notion of measure space

Definition 13.
If $\mathscr{A}$ is a σ-algebra on the universe X, and if μ is a measure on $\mathscr{A}$, the triple $(X, \mathscr{A}, \mu)$ is called a measure space.

Examples of measures

Example 14.
On any given measurable space $(X, \mathscr{A})$ we fix an element $a \in X$, and we consider the corresponding *Dirac*-measure $\mu : \mathscr{A} \to [0, +\infty]$ defined by

$$\mu(A) := \begin{cases} 1 & \text{, if } a \in A \\ 0 & \text{, otherwise.} \end{cases}$$

Example 15.
On any given measurable space $(X, \mathscr{A})$ we can consider the measure $\mu : \mathscr{A} \to [0, +\infty]$ defined by

$$\mu(A) := |A| = \begin{cases} |A| \text{ i.e. the number of elements in the set } A & \text{, if } A \text{ is finite} \\ +\infty & \text{, otherwise.} \end{cases}$$

We will use this measure on universes like $\mathbb{N}$ or $\mathbb{Z}$. On the universe $\mathbb{R}$, this measure does not generalize the concept of length.

Constructing measures is far to be trivial

The question, how to define a measure on a σ-algebra that contains "usual subsets" as rectangles cannot be solved in an easy way as in the above examples. The construction of such measures is in fact, as we will see, "far to be trivial".

The notion of intervals in $\mathbb{R}^p$

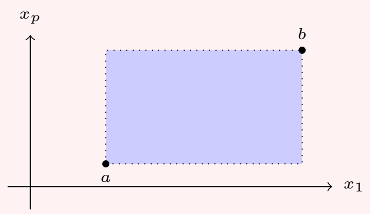
Definition 16.

For given vectors $a = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix} \in \mathbb{R}^p$ $(p = 1, 2, 3, \ldots)$ we put

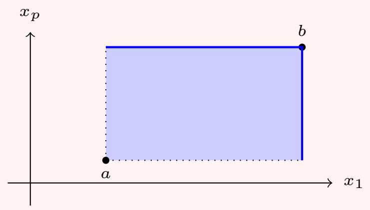
1. $a < b$ iff:
$$\forall k = 1, 2, \ldots, p, \quad a_k < b_k$$

2. $a \leq b$ iff:
$$\forall k = 1, 2, \ldots, p, \quad a_k \leq b_k$$

3. open interval $]a, b[$:
$$]a, b[:= \begin{cases} \{x \in \mathbb{R}^p \ : \ a_k < x_k < b_k \ : \ k = 1, 2, \ldots, p\} & \text{, if } a < b \\ \varnothing & \text{, otherwise} \end{cases}$$

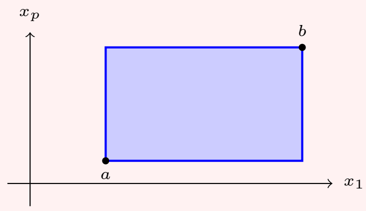


4. semi-open interval $]a, b]$:
$$]a, b] := \begin{cases} \{x \in \mathbb{R}^p \ : \ a_k < x_k \leq b_k \ : \ k = 1, 2, \ldots, p\} & \text{, if } a < b \\ \varnothing & \text{, otherwise} \end{cases}$$



5. closed interval $[a, b]$:

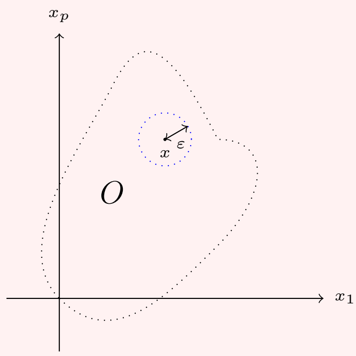
$$[a, b] := \begin{cases} \{x \in \mathbb{R}^p \ : \ a_k \leq x_k \leq b_k \ : \ k = 1, 2, \ldots, p\} & \text{, if } a \leq b \\ \varnothing & \text{, otherwise} \end{cases}$$



Open sets in $\mathbb{R}^p$

Definition 17.
A subset O of $\mathbb{R}^p$ ($p = 1, 2, 3, \ldots$) is called an open set iff

$$\forall x \in O, \exists \varepsilon > 0 \text{ such that } \{y \in \mathbb{R}^p \ : \ \|y - x\| < \varepsilon\} \subset O.$$



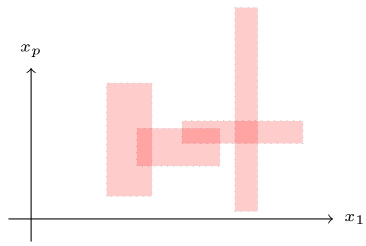
A typical open set in $\mathbb{R}^p$

A typical example in $\mathbb{R}$ for an open set is an open interval

$$O =]a, b[\qquad (a < b)$$

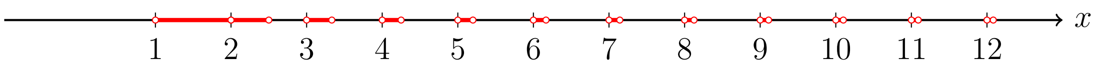
or a union of such intervals

$$O = \bigcup_{k=1}^{n}]a_k, b_k[\qquad \text{or} \qquad O = \bigcup_{k=1}^{+\infty}]a_k, b_k[$$



In $\mathbb{R}$, an example of such an union would be the set

$$O = \bigcup_{n=1}^{\infty}]n, n + \frac{1}{n}[.$$



Remark that the empty set $\varnothing$ is an open set, as well as $\mathbb{R}^p$.

The family of open sets

Definition 18.
We denote by $\mathscr{O}^p$ the collection of all open sets in $\mathbb{R}^p$ (for $p = 1, 2, 3, \ldots$).

Proposition 19.
The family $\mathscr{O}^p$ of open sets in $\mathbb{R}^p$ ($p = 1, 2, 3, \ldots$) is not *a σ-algebra.*

Proof. Consider the open sets $]1, 2 + \frac{1}{n}[$ (for $n = 1, 2, 3 \ldots$) in $\mathbb{R}$. Then

$$\bigcap_{n=1}^{+\infty}]1, 2 + \frac{1}{n}[=]1, 2]$$

is not open. Thus, $\mathscr{O}^p$ is not $\bigcap_{n=1}^{+\infty}$-stable. □

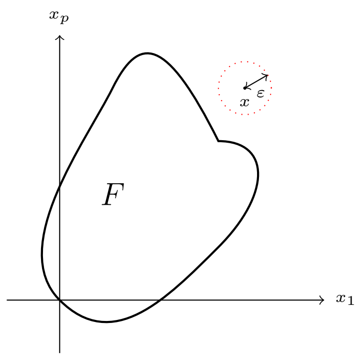
A final remark on open sets

A measure that extends the notion of length, area or volume should be defined on a σ-algebra containing the family of open sets $\mathscr{O}^p$.

Closed sets in $\mathbb{R}^p$

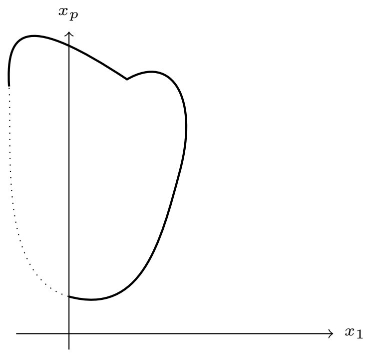
Definition 20.
A subset F of $\mathbb{R}^p$ (for $p = 1, 2, 3, \ldots$) is a <u>closed</u> set iff

$$\complement F \in \mathscr{O}^p, \qquad \text{i.e. its complement is open.}$$



Remark 21. *A set $A \subset \mathbb{R}^p$ can be*

- *open and closed at the same time: as an example take $A = \mathbb{R}$;*
- *neither open nor closed: as an example take $]1, 2]$ in $\mathbb{R}$.*



The family of closed sets

Definition 22.
We denote by $\mathscr{F}^p$ the collection of all closed sets in $\mathbb{R}^p$ (for $p = 1, 2, 3, \ldots$).

Proposition 23.
The family $\mathscr{F}^p$ of closed sets in $\mathbb{R}^p$ ($p = 1, 2, 3, \ldots$) is not *a σ-algebra.*

Proof. Consider the closed sets $[1, 2 - \frac{1}{n}]$ (for $n = 1, 2, 3 \ldots$) in $\mathbb{R}$. Then

$$\bigcup_{n=1}^{+\infty} [1, 2 - \frac{1}{n}] = [1, 2[$$

is not closed. Thus, $\mathscr{F}^p$ is not $\bigcup_{n=1}^{+\infty}$-stable. □

A final remark on closed sets

A measure that extends the notion of length, area or volume should be defined on a σ-algebra containing the family of open sets $\mathscr{O}^p$ as well as the family of closed sets $\mathscr{F}^p$.

The family of semi-open intervals

Definition 24.
We denote by $\mathscr{I}^p$ the collection of all semi-open sets of the form]a,b] in $\mathbb{R}^p$ (for $p = 1, 2, 3, \ldots$).
Remark that $\varnothing \in \mathscr{I}^p$.

Proposition 25.
The family $\mathscr{I}^p$ of semi-open intervals in $\mathbb{R}^p$ ($p = 1, 2, 3, \ldots$) is not *a σ-algebra.*

Proof. Consider the semi-open $]1 - \frac{1}{n}, 2]$ (for $n = 1, 2, 3 \ldots$) in $\mathbb{R}$. Then

$$\bigcap_{n=1}^{+\infty}]1 - \frac{1}{n}, 2] = [1, 2]$$

is not semi-open. Thus, $\mathscr{I}^p$ is not $\bigcap_{n=1}^{+\infty}$-stable. □

A final remark on semi-open intervals

The notion of length, area and volume is well-defined on semi-open intervals. Thus, a measure that extends these notions should be defined on a σ-algebra containing the family of semi-open intervals $\mathscr{I}^p$ as well as the families $\mathscr{O}^p$ and $\mathscr{F}^p$.

Different generators for the σ-algebra of interest

As yet mentioned, we want to extend the notion of length, area and volume of simple geometric sets to a family $\mathscr{B}(\mathbb{R}^p)$ consisting of more complex subsets of $\mathbb{R}^p$ ($p = 1, 2, 3, \ldots$).
The family of open sets $\mathscr{O}^p$ should be contained in $\mathscr{B}(\mathbb{R}^p)$. Thus we define

$$\mathscr{B}(\mathbb{R}^p) := \sigma(\mathscr{O}^p).$$

Definition 26.
The σ-algebra $\mathscr{B}(\mathbb{R}^p)$ generated by the family of open sets $\mathscr{O}^p$ in $\mathbb{R}^p$ is called the Borel-algebra.

Proposition 27.
The Borel-algebra $\mathscr{B}(\mathbb{R}^p)$ ($p = 1, 2, 3, \ldots$) is generated by

- *the family of open sets $\mathscr{O}^p$,*
- *the family of closed sets $\mathscr{F}^p$ as well as*
- *the family of semi-open sets $]a, b]$ contained in $\mathscr{J}^p$.*

Thus

$$\boxed{\mathscr{B}(\mathbb{R}^p) = \sigma(\mathscr{O}^p) = \sigma(\mathscr{F}^p) = \sigma(\mathscr{J}^p).}$$

The proof of these facts relies on the monotonicity of the $\sigma(\cdot)$-operator:

$$\mathscr{E}_1 \subset \sigma(\mathscr{E}) \Longrightarrow \sigma(\mathscr{E}_1) \subset \sigma(\mathscr{E}).$$

Proof. **Step 1:** $\sigma(\mathscr{O}^p) = \sigma(\mathscr{F}^p)$
Since the σ-algebra $\sigma(\mathscr{O}^p)$ is $\complement$-stable and since $\complement\mathscr{F}^p = \mathscr{O}^p$, we have $\mathscr{F}^p \subset \sigma(\mathscr{O}^p)$; thus

$$\sigma(\mathscr{F}^p) \subset \sigma(\mathscr{O}^p).$$

Since the σ-algebra $\sigma(\mathscr{F}^p)$ is $\complement$-stable and since $\complement\mathscr{O}^p = \mathscr{F}^p$, we have $\mathscr{O}^p \subset \sigma(\mathscr{F}^p)$; thus

$$\sigma(\mathscr{O}^p) \subset \sigma(\mathscr{F}^p).$$

Thus we may conclude that $\sigma(\mathscr{O}^p) = \sigma(\mathscr{F}^p) = \mathscr{B}(\mathbb{R}^p)$.
Step 2: $\sigma(\mathscr{O}^p) = \sigma(\mathscr{J}^p)$
Since any semi-open interval $]a, b]$ with $a < b$ can be written as

$$]a, b] = \bigcap_{n=1}^{\infty} \left]a, b + \frac{1}{n}\right[,$$

we have $\mathscr{J}^p \subset \sigma(\mathscr{O}^p)$; thus $\sigma(\mathscr{J}^p) \subset \sigma(\mathscr{O}^p)$.
If we can show that $\mathscr{O}^p \subset \sigma(\mathscr{J}^p)$, we may conclude that $\sigma(\mathscr{O}^p) \subset \sigma(\mathscr{J}^p)$, so that $\sigma(\mathscr{J}^p) = \sigma(\mathscr{O}^p)$.
Thus it remains to show that $\mathscr{O}^p \subset \sigma(\mathscr{J}^p)$.

Let us first recall that any open set in $\mathbb{R}^p$ may be written as a union of open intervals:

$$\forall O \in \mathscr{O}^p, \quad O = \bigcup_{i \in I}]a_i, b_i[, \qquad \text{where } I \text{ is finite or countable.}$$

Any open interval $]a, b[$ can be written as a countable union of semi-open intervals:

$$]a, b[= \bigcup_{n=1}^{\infty}]a, b - 1/n].$$

Hence the family of open intervals is contained in $\sigma(\mathscr{J}^p)$, and the above remark implies now that

$$\mathscr{O}^p \subset \sigma(\mathscr{J}^p) \qquad \text{and} \qquad \sigma(\mathscr{O}^p) \subset \sigma(\mathscr{J}^p).$$

□

1.2. How to define measures on $\mathbb{R}^p$

Our starting point

First of all, we construct a so-called pre-measure on $\mathscr{J}^p$, i.e. a mapping

$$\mu : \mathscr{J}^p \to [0, +\infty], \quad A \mapsto \mu(A)$$

with the following properties:

1. $\mu(\varnothing) = 0$;
2. μ is *additive*:

$$\left.\begin{array}{l} A_1, \ldots, A_n \in \mathscr{J}^p, \\ \quad \text{pairwise disjoint} \\ \dot{\bigcup}_{k=1}^n A_k \in \mathscr{J}^p \end{array}\right\} \Longrightarrow \mu\left(\dot{\bigcup_{k=1}^n} A_k\right) = \sum_{k=1}^n \mu(A_k)$$

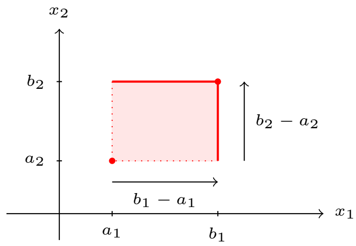
3. μ is σ-*additive*:

$$\left.\begin{array}{l} \{A_k\}_{k=1}^{+\infty} \text{ in } \mathscr{J}^p, \\ \quad \text{pairwise disjoint} \\ \dot{\bigcup}_{k=1}^\infty A_k \in \mathscr{J}^p \end{array}\right\} \Longrightarrow \mu\left(\dot{\bigcup_{k=1}^\infty} A_k\right) = \sum_{k=1}^\infty \mu(A_k)$$

For $p = 1, 2, 3, \ldots$, we can define a pre-measure by

$$\mu(]a, b]) = (b_1 - a_1) \cdot (b_2 - a_2) \cdot \cdots \cdot (b_p - a_p) \qquad \text{(for all } a \leq b\text{)}.$$

This is the usual length, area or volume.



Definition 28.
We call this pre-measure the Lebesgue-pre-measure.

For $p = 1$, we may consider a function $f : \mathbb{R} \to \mathbb{R}$ that is

- Monotonically non-decreasing (we denote this by $f \nearrow$):

$$\forall x_1, x_2 \in \mathbb{R}, \quad x_1 < x_2 \Longrightarrow f(x_1) \le f(x_2).$$

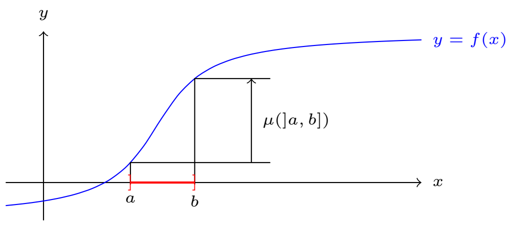
- Right-continuous:

$$\forall \xi \in \mathbb{R}, \qquad \lim_{x \to \xi^+} f(x) = f(\xi).$$

(As an example one may take $f(x) \equiv x$.) Then

$$\mu_f(]a, b]) = f(b) - f(a) \qquad \text{for } a \le b$$

is a pre-measure on $\mathscr{J}^1$.



Definition 29.
We call this measure a Stieltjes-Lebesgue-pre-measure.

Remark 30. *Remark that for $f(x) \equiv x$, this measure is in fact the Lebesgue-pre-measure $\mu(]a, b]) = b - a \quad$ (for $a \le b$).*

Remark 31. *The fact that the above defined (Stieltjes-)-Lebesgue-pre-measures are positive and additive can be proven in an easy way.*

The proof of the σ-additivity is rather technical: for $p = 1$, this proof heavily relies on the right-continuity of the generating function f and on the Heine-Borel Lemma.

First step: extension by additivity

We consider the family $\mathscr{R}$ consisting of finite unions of semi-open intervals $]a, b]$. Thus, any element of $\mathscr{R}$ is of the form

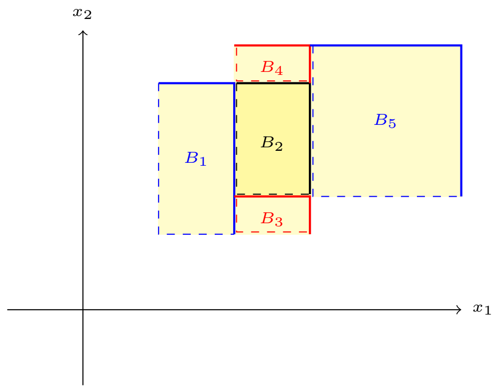
$$A = \bigcup_{k=1}^{m} A_k \qquad \text{, where } m \text{ is a natural number and } A_k \in \mathscr{I}^p (k = 1, 2, \ldots, m).$$

Clearly $\mathscr{I}^p \subset \mathscr{R}$.

Moreover, any such element can be written as a disjoint union of semi-open intervals belonging to $\mathscr{I}^p$:

$$A = \dot{\bigcup_{k=1}^{n}} B_k \qquad \text{, where } n \text{ is a natural number and } B_k \in \mathscr{I}^p (k = 1, 2, \ldots, n).$$

This can be illustrated in the following way:



For each subset $A \in \mathscr{R}$ written as a disjoint union $\dot\bigcup_{k=1}^{n} B_k$ of element $B_k \in \mathscr{I}^p$ we put

$$\tilde{\mu}(\dot{\bigcup_{k=1}^{n}} B_k) = \sum_{k=1}^{n} \mu(B_k).$$

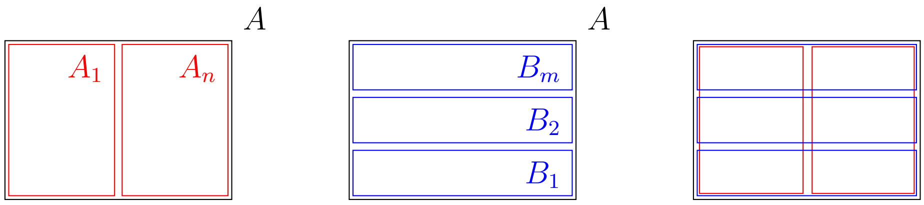
We obtain in this way an extension of μ to $\mathscr{R}$. Remark that the term extension means the following

$$\forall A \in \mathscr{I}^p, \quad \tilde{\mu}(A) = \tilde{\mu}(A \,\dot\cup\, \varnothing) = \mu(A) + \mu(\varnothing) = \mu(A).$$

Moreover, the definition of $\tilde{\mu}$ does not depend on the chosen disjoint union, i.e.

$$\left.\begin{array}{l} \dot\bigcup_{k=1}^{n} A_k = \dot\bigcup_{j=1}^{m} B_j \\ A_k, B_j \in \mathscr{I}^p \\ (k = 1, \ldots, n, j = 1, \ldots, m) \end{array}\right\} \Longrightarrow \sum_{k=1}^{n} \mu(A_k) = \sum_{j=1}^{m} \mu(B_j)$$

This can be illustrated in the following way



The proof relies on the above illustration:

$$\begin{aligned} \mu(A_k) &= \mu\left(\dot{\bigcup_{j=1}^{m}}(A_k \cap B_j)\right) = \sum_{j=1}^{m} \mu(A_k \cap B_j) \\ \sum_{k=1}^{n} \mu(A_k) &= \sum_{k=1}^{n}\sum_{j=1}^{m} \mu(A_k \cap B_j) \\ &= \ldots = \sum_{j=1}^{m} \mu(B_j) \end{aligned}$$

Proposition 32.
The above extension

$$\tilde{\mu} : \mathscr{R} \to [0, +\infty], \quad A \mapsto \tilde{\mu}(A)$$

is a pre-measure on $\mathscr{R}$. Thus

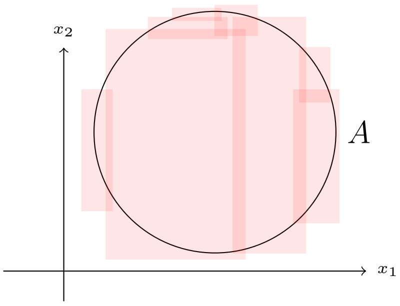
$$\left.\begin{array}{l} \{A_k\}_{k=1}^{+\infty} \text{ in } \mathscr{R}, \\ \quad \text{pairwise disjoint} \\ \dot{\bigcup}_{k=1}^{\infty} A_k \in \mathscr{R} \end{array}\right\} \Longrightarrow \tilde{\mu}\left(\dot{\bigcup_{k=1}^{\infty}} A_k\right) = \sum_{k=1}^{\infty} \tilde{\mu}(A_k)$$

Second step: the external measure

We put, $\forall A \subset \mathbb{R}$,

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \tilde{\mu}(A_n) \ : \ A_n \in \mathscr{R}, A \subset \bigcup_{n=1}^{\infty} A_n\right\}$$

(this sum can be finite by choosing $A_n = \varnothing$ for all except a finite number of elements A_n).



Definition 33.
This approximation from outside is called an <u>external measure</u>.

The main property of an external measure is that, in general, it is *not* a measure! We have

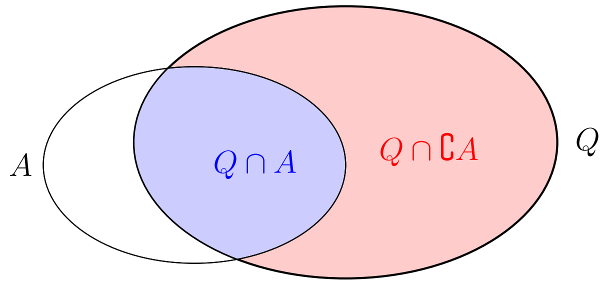
Proposition 34.

- $\mu^*(\varnothing) = 0$;
- $A \subset B \Longrightarrow \mu^*(A) \leq \mu^*(B)$;
- $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$;
- μ^* *is an extension of* μ, *i.e.* $A \in \mathscr{J}^p \Longrightarrow \mu^*(A) = \mu(A)$;
- *But in general,* μ^* *is not even additive (and thus not* σ*-additive)!*

Third and last step: good sub-dividers

In a third and last step, we call a subset $A \subset \mathbb{R}^p$ μ^**-measurable* or a *good sub-divider with respect to* μ^* if

$$\forall Q \subset \mathbb{R}^p, \qquad \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \cap \complement A).$$



We collect all μ^*-measurable sets (the good sub-dividers) in a family:

$$\mathscr{A}_\mu := \{A \subset \mathbb{R}^p \;:\; \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \cap \complement A), \quad \forall Q \subset \mathbb{R}^p\}$$

and we put

$$\hat{\mu}(A) := \mu^*(A), \qquad \forall A \in \mathscr{A}_\mu.$$

It turns out that $\mathscr{A}_\mu$ is a σ-algebra containing $\mathscr{J}^p$ and that $\hat{\mu}$ is a measure!

Proposition 35.

1. $\mathscr{A}_\mu$ *is a* σ*-algebra containing* $\mathscr{J}^p$; *hence*

$$\mathscr{B}(\mathbb{R}^p) \subset \mathscr{A}_\mu.$$

2. $\hat{\mu} : \mathscr{A}_\mu \to [0, +\infty]$ *is a measure extending the pre-measure* μ*:*

$$\forall\,]a,b] \in \mathscr{J}^p, \qquad \hat{\mu}(]a,b]) = \mu(]a,b]).$$

Definition 36.
If the starting pre-measure μ is the Lebesgue-pre-measure

$$\mu(]a,b]) = \prod_{k=1}^{p}(b_k - a_k) = (b_1 - a_1) \cdot \dots \cdot (b_p - a_p), \qquad \text{for all } a \leq b \in \mathbb{R}^p,$$

we denote the so obtained σ-algebra $\mathscr{A}_\mu$ by $\mathscr{L}^p$ and the so obtained measure $\hat{\mu}$ by λ^p.
If $p = 1$ we may replace $\mathscr{L}^1$ by $\mathscr{L}$ and λ^1 by λ.
The measure λ^p is called the Lebesgue-measure on $\mathbb{R}^p$.
If the starting pre-measure is a Stieltjes-Lebesgue-pre-measure μ_f, the extensions $\hat{\mu}$ are called Stieltjes-Lebesgue-measures on $\mathbb{R}$.

Remark 37. *Two questions remain open:*

1. *Is the extension $\hat{\mu}$ of μ unique?*

2. *How much bigger than $\mathscr{B}(\mathbb{R}^p)$ is $\mathscr{A}_\mu$ (resp $\mathscr{L}^p$)?*

We will address the second question later. Concerning the first question, we can give the following result.

Proposition 38.
The extension of the pre-measure

$$\mu : \mathscr{J}^p \to [0, +\infty], \quad]a,b] \mapsto \mu(]a,b])$$

to the measure

$$\hat{\mu} : \mathscr{A}_\mu \to [0, +\infty], \quad A \mapsto \hat{\mu}(A)$$

is unique if the starting pre-measure μ is σ-finite, i.e. if

$$\exists \{E_n\}_{n=1}^{+\infty} \text{ in } \mathscr{J}^p \text{ such that } \mathbb{R}^p = \bigcup_{n=1}^{\infty} E_n \text{ and } \mu(E_n) < \infty \text{ for all } n.$$

Corollary 39.
The extension of the Lebesgue-pre-measure

$$\mu : \mathscr{J}^p \to [0, +\infty], \quad]a, b] \mapsto \mu(]a, b]) = \prod_{k=1}^{n} (b_k - a_k)$$

to the Lebesgue-measure

$$\lambda^p : \mathscr{L}^p \to [0, +\infty], \quad A \mapsto \lambda^p(A)$$

is unique.

Corollary 40.
The extension of a Stieltjes-Lebesgue-pre-measure

$$\mu_f : \mathscr{J}^p \to [0, +\infty], \quad]a, b] \mapsto \mu(]a, b]) = f(b) - f(a)$$

(for example $\mu(]a, b]) = f(b) - f(a)$ if $p = 1$) to the Stieltjes-Lebesgue-measure

$$\hat{\mu} : \mathscr{A}_\mu \to [0, +\infty], \quad A \mapsto \hat{\mu}(A)$$

is unique.

1.3. Properties of measures

Monotonicity of measures

Proposition 41.

Hyp: $\mu : \mathscr{A} \to [0, +\infty]$ *a measure defined on a* σ*-algebra* $\mathscr{A}$.
Concl: μ *is monotonous: for all* A_1, $A_2 \in \mathscr{A}$,

$$A_1 \subset A_2 \Longrightarrow \mu(A_1) \leq \mu(A_2).$$

Proof. We can write

$$A_2 = A_1 \mathbin{\dot\cup} (A_2 \setminus A_1) \qquad \text{with } A_2 \setminus A_1 \in \mathscr{A}.$$

Thus, by additivity,

$$\mu(A_2) = \mu(A_1) + \mu(A_2 \setminus A_1) \geq \mu(A_1).$$

□

Subtractivity of a measure

Proposition 42.

Hyp $\mu : \mathscr{A} \to [0, +\infty]$ *a measure defined on a* σ*-algebra* $\mathscr{A}$.
Concl *For all* A_1 *and* $A_2 \in \mathscr{A}$,

$$\left.\begin{array}{l} A_1 \subset A_2 \\ \mu(A_1) < +\infty \end{array}\right\} \Longrightarrow \mu(A_2 \setminus A_1) = \mu(A_2) - \mu(A_1).$$

Proof. This follows from

$$A_2 = A_1 \cup (A_2 \setminus A_1)$$

and

$$\mu(A_2) = \underbrace{\mu(A_1)}_{<+\infty} + \mu(A_2 \setminus A_1) \qquad \text{i.e.} \qquad \mu(A_2 \setminus A_1) = \mu(A_2) - \mu(A_1).$$

□

Generalized additivity

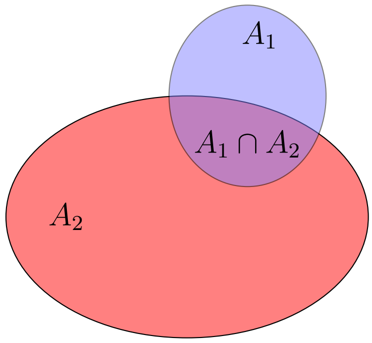
Proposition 43.

Hyp $\mu : \mathscr{A} \to [0, +\infty]$ *a measure defined on a* σ*-algebra* $\mathscr{A}$.
Concl *For all* A_1 *and* $A_2 \in \mathscr{A}$,

$$\mu(A_1) + \mu(A_2) = \mu(A_1 \cup A_2) + \mu(A_1 \cap A_2)$$

and hence, if $\mu(A_1 \cap A_2) < +\infty$,

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2).$$



Proof. We have

$$\mu(A_1 \cup A_2) = \mu(A_1 \setminus (A_1 \cap A_2)) + \mu(A_2)$$

and thus

$$\begin{aligned} \mu(A_1 \cup A_2) &+ \mu(A_1 \cap A_2) = \\ &= \mu(A_1 \setminus (A_1 \cap A_2)) + \mu(A_1 \cap A_2) + \mu(A_2) \\ &= \mu(A_1) + \mu(A_2). \end{aligned}$$

□

Continuity of measures

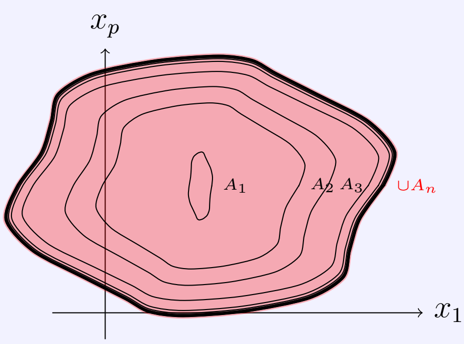
Proposition 44.

Hyp $\mu : \mathscr{A} \to [0, +\infty]$ *a measure defined on a* σ*-algebra* A.

1. μ *is continuous from below, i.e. for all non-decreasing* $\{A_n\}_{n=1}^{+\infty}$ *in* $\mathscr{A}$ *(this means* $A_n \subset A_{n+1}$ *for* $n = 1, 2, 3, \ldots$ *and we write* $A_n \nearrow$*) we have*

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

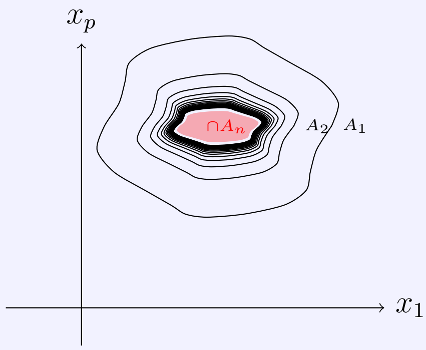
Remark that the sequence $\{\mu(A_n)\}_{n=1}^{+\infty}$ *is non-decreasing.*



2. *μ is continuous from above, i.e. for all non-increasing $\{A_n\}_{n=1}^{+\infty}$ in $\mathscr{A}$ (this means $A_n \supset A_{n+1}$ for $n = 1, 2, 3, \ldots$ and we write $A_n \searrow$) with $\mu(A_1) < +\infty$ we have*

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right).$$

Remark that the sequence $\{\mu(A_n)\}_{n=1}^{+\infty}$ is non-increasing.



3. *μ is continuous at $\varnothing$, i.e. for all non-increasing $\{A_n\}_{n=1}^{+\infty}$ in $\mathscr{A}$ (this means $A_n \supset A_{n+1}$ for $n = 1, 2, 3, \ldots$) with $\mu(A_1) < +\infty$ and $\bigcap_{n=1}^{\infty} A_n = \varnothing$ we have*

$$\lim_{n\to\infty} \mu(A_n) = \mu(\varnothing) = 0.$$

Remark that the sequence $\{\mu(A_n)\}_{n=1}^{+\infty}$ is non-increasing.

1.4. How much is $\mathscr{A}_\mu$ bigger than $\mathscr{B}(\mathbb{R}^p)$?

The extension of a pre-measure μ given on $\mathscr{J}^p$ gives a measure defined on a σ-algebra $\mathscr{A}_\mu$:

$$\mathscr{J}^p \subset \sigma(\mathscr{J}^p) = \mathscr{B}(\mathbb{R}^p) \subset \mathscr{A}_\mu.$$

Now we address the question of how much bigger the σ-algebra $\mathscr{A}_\mu$ is in comparison with the smallest σ-algebra containing $\mathscr{J}^p$.

We will see that, in general, $\mathscr{A}_\mu$ is bigger than $\sigma(\mathscr{J}^p)$, but just by a small, useful amount:

$$\sigma(\mathscr{J}^p) = \mathscr{B}(\mathbb{R}^p) \subsetneq A_\mu \subsetneq \mathscr{P}(\mathbb{R}^p).$$

In order to describe the excess of $\mathscr{A}_\mu$ on $\sigma(\mathscr{J}^p)$, we need two new concepts: the concept of *null-sets* and that of *complete measures*.

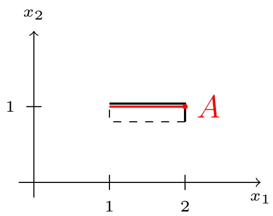
Null-sets

Definition 45.
Given: a measure space $(X, \mathscr{A}, \mu)$
we say: $A \in \mathscr{A}$ is a μ-null-set iff:

$$\mu(A) = 0.$$

Example 46.
A non-trivial example of a null-set is given in the measure space $(\mathbb{R}^2, \mathscr{L}^2, \lambda^2)$ by

$$A = \{(x, y) \in \mathbb{R} \; : \; y = 1, 1 < x \leq 2\}.$$



Proof. Remark that $A \in \mathscr{L}^2$, since A is an intersection of semi-open intervals:

$$A = \bigcap_{n=1}^{\infty}](1, 1 - 1/n), (2, 1)]$$

The claim now follows from

$$\lambda^2(A) = \lim_{n\to\infty} (2 - 1) \times (1 - (1 - 1/n)) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

□

Completeness of measures

Definition 47.
Given: A measure space $(X, \mathscr{A}, \mu)$
we say: X is complete iff:
every subset of a μ-null-set is a μ-null-set, too, i.e. if

$$\left.\begin{array}{l} A \in \mathscr{A} \text{ with } \mu(A) = 0 \\ B \subset A \end{array}\right\} \Longrightarrow B \in \mathscr{A} \text{ and } \mu(B) = 0.$$

Remark 48. *Working with a non-complete measure $\mu : \mathscr{A} \to [0, +\infty]$ leads to counter-intuitive situations. As an example, if A_1 and A_2 are to subsets such that*

$$A_1 \subsetneq A_2, \quad \mu(A_1) = \mu(A_2) < +\infty,$$

the null-set $A_2 \setminus A_1$ could contain a subset E with the following properties

$$A_1 \subsetneq A_1 \uplus E \subsetneq A_2, \quad \mu(A_1) = \mu(A_2), \qquad \text{but } \mu(A_1 \uplus E) \text{ is not defined!}$$

Hence, it is a good idea to avoid non-complete measures.

The completeness of the Lebesgue-Stieltjes measures

Proposition 49.
The following measures are complete:

1. *The Lebesgue-Stieltjes measures*

$$\mu_f : \mathscr{A}_{\mu_f} \to [0, +\infty]$$

corresponding to a monotonically non-decreasing, right-continuous function $f : \mathbb{R} \to \mathbb{R}$.
It is the smallest complete extension of the pre-measure μ_f on $\mathscr{J}^1$.

2. *The Lebesgue measures*

$$\lambda^p : \mathscr{L}^p \to [0, +\infty]$$

for $p = 1, 2, 3, , \ldots$.
They are the smallest complete extension of the pre-measure λ^p on $\mathscr{J}^p$.

The starting point:

The Lebesgue-pre-measure: $\lambda^p : \mathscr{J}^p \to [0, +\infty]$

$]a, b] \mapsto \lambda^p(]a, b]) = \prod_{k=1}^{p}(b_k - a_k)$

↓ unique extension

The Lebesgue-measure: $\lambda^p : \mathscr{L}^p \to [0, +\infty]$

$A \mapsto \lambda^p(A)$ with $\mathscr{B}(\mathbb{R}^p) \subsetneq \mathscr{L}^p$

↓ restriction

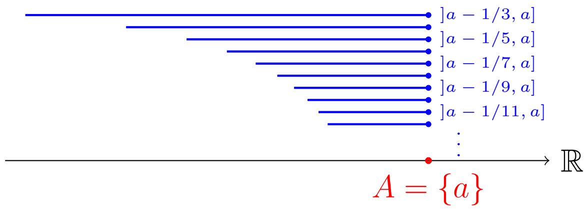
$\beta^p := \lambda|_{\mathscr{B}(\mathbb{R}^p)} : \mathscr{B}(\mathbb{R}^p) \to [0, +\infty]$ is *not* complete (as an other intermediate restriction)

Examples of λ^1-null-sets

Example 50.
Let $a \in \mathbb{R}$ be fixed and consider the singleton $A := \{a\}$. Then

1. $A \in \mathscr{L}^1$ since $\{a\} = \bigcap_{n=1}^{+\infty}]a - 1/n, a]$ and $]a - 1/n, a] \in \mathscr{J}^1$.
2. A is a λ^1-null-set since (by continuity from above)

$$\lambda^1(\{a\}) = \lambda^1(\bigcap_{n=1}^{+\infty}]a - 1/n, a]) = \lim_{n\to\infty} \lambda^1(]a - 1/n, a]) = \lim_{n\to\infty} \frac{1}{n} = 0.$$



Example 51.
Let $a_1,\ a_2,\ \ldots a_n$ be n different real numbers, and consider the (finite) set $A := \{a_1, a_2, \ldots, a_n\}$. Then

1. $A \in \mathscr{L}^1$ since $A = \{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_n\}$ and $\{a_k\} \in \mathscr{L}^1$ (for $k = 1, 2, \ldots, n$).
2. A is a λ^1-null-set since, by additivity,

$$\lambda^1(A) = \lambda^1\left(\dot{\bigcup_{k=1}^{n}}\{a_k\}\right) = \sum_{k=1}^{n} \lambda^1(\{a_k\}) = \sum_{k=1}^{n} 0 = 0.$$

Example 52.
Let a_1, a_2, a_3, ... be a sequence of different real numbers, and consider the (infinite) set $A := \{a_1, a_2, a_3, \ldots\}$. Then

1. $A \in \mathscr{L}^1$ since $A = \{a_1\} \cup \{a_2\} \cup \cdots = \dot\bigcup_{k=1}^{+\infty} \{a_k\}$ and $\{a_k\} \in \mathscr{L}^1$ (for $k = 1, 2, \ldots$).
2. A is a λ^1-null-set since, by σ-additivity,

$$\lambda^1(A) = \lambda^1\left(\dot\bigcup_{k=1}^{+\infty} \{a_k\}\right) = \sum_{k=1}^{+\infty} \lambda^1(\{a_k\}) = \sum_{k=1}^{+\infty} 0 = 0.$$

Thus for example, $\mathbb{Q}$ (as a countable set) is a λ^1-null-set:

$$\mathbb{Q} \in \mathscr{L}^1 \qquad \text{and} \qquad \lambda^1(\mathbb{Q}) = 0.$$

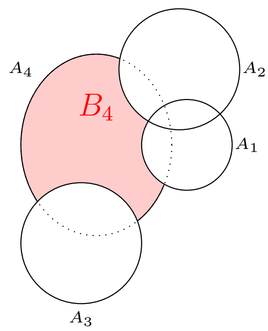
Countable unions of null-sets

Proposition 53.

Hyp
- *$(X, \mathscr{A}, \mu)$ a measure space*
- *$\{A_n\}_{n=1}^{+\infty}$ is a sequence of μ-null-sets in X.*

Concl *The countable union $\bigcup_{n=1}^{\infty} A_n$ is a μ-null-set, too:*

$$\mu(A_n) = 0 \text{ for } n = 1, 2, 3, \ldots \quad \Longrightarrow \quad \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$$



Proof. Put $B_1 := A_1$ and $B_n := A_n \setminus \bigcup_{k=1}^{n-1} A_k$. Then

- $B_n \in \mathscr{A}$ for $n = 1, 2, 3, \ldots$ and $B_n \subset A_n$. Thus, by monotonicity, $\mu(B_n) = 0$ for $n = 1, 2, 3 \ldots$.
- $\bigcup_{n=1}^{\infty} A_n = \dot{\bigcup}_{n=1}^{\infty} B_n$, so that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\dot{\bigcup_{n=1}^{\infty}} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} 0 = 0.$$

□

Almost everywhere true properties

We collect in a set E all the points where a given property is true:

$$E := \{x \in X \ : \ \text{this property is true at } x\}.$$

Definition 54.
We say that this property holds *μ-almost everywhere (or in short μ-a.e.)* if and only if

$$\exists N \in \mathscr{A} \text{ with } \complement E \subset N \text{ and } \mu(N) = 0.$$

Remark 55. *Remark that if the measure μ is complete, one may take*

$$N = \complement E$$

and we have

$$\mu(\complement E) = 0.$$

Example 56.
The function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

defined on the measure space $(\mathbb{R}, \mathscr{L}, \lambda^1)$ is λ^1-almost everywhere zero. We write

$$f = 0 \qquad \lambda^1\text{-a.e.}$$

Indeed, we have

$$E := \{x \in \mathbb{R} \ : \ f(x) = 0\} = \complement\mathbb{Q}$$

and

$$\lambda^1(\complement E) = \lambda^1(\mathbb{Q}) = 0.$$

Remark 57. *Let us point out that*

$$E = f^{-1}(\{0\}) := \{x \in \mathbb{R} \ : \ f(x) = 0\}$$

is a pre-image of a singleton.

1.5. Measurable functions

Pre-images of mappings

For any given mapping

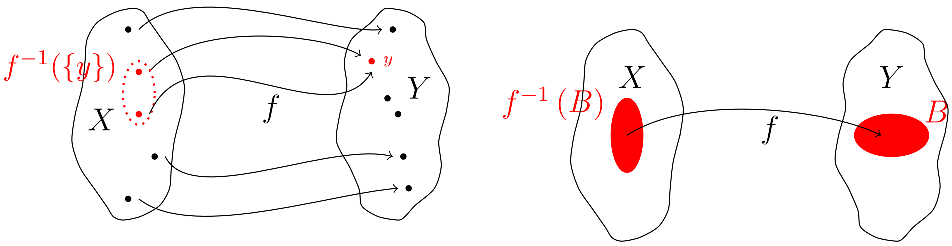
$$f : X \to Y, \quad x \mapsto y := f(x)$$

one may consider the pre-images

$$f^{-1}(\{y\}) := \{x \in X \ : \ f(x) = y\}, \qquad \forall y \in Y$$

or more generally

$$f^{-1}(B) := \{x \in X \ : \ f(x) \in B\}, \qquad \forall B \subset Y.$$



Pre-images have nice properties as

$$\begin{aligned} f^{-1}\left(\bigcup_{\iota\in I} B_\iota\right) &= \bigcup_{\iota\in I} f^{-1}(B_\iota) \\ f^{-1}\left(\bigcap_{\iota\in I} B_\iota\right) &= \bigcap_{\iota\in I} f^{-1}(B_\iota) \\ f^{-1}(\complement B) &= \complement f^{-1}(B)\,. \end{aligned}$$

Thereby I may be any index set as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, ...

An important property of pre-images

If $\mathscr{B}$ is a family of subsets of Y, we put

$$f^{-1}(\mathscr{B}) := \{ \underbrace{f^{-1}(B)}_{:=\{x\in X\,:\,f(x)\in B\}} \ : \ B \in \mathscr{B}\}.$$

Hyp Suppose that

- $\mathscr{B}$ is a σ-algebra over Y,
- that this σ-algebra is generated by $\mathscr{E}$, i.e. $\mathscr{B} = \sigma(\mathscr{E})$, and that
- $f : X \to Y$ is a mapping

Concl We have

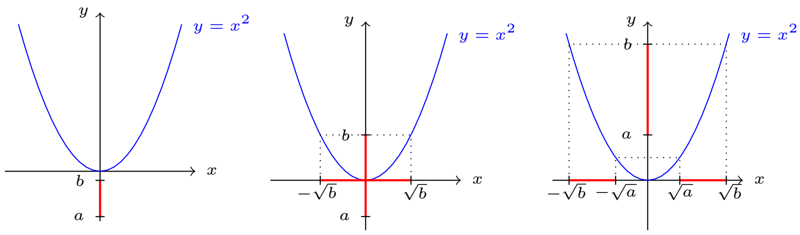
$$\boxed{\sigma\left(f^{-1}(\mathscr{E})\right) = f^{-1}(\sigma(\mathscr{E}))\,.}$$

Pre-images under continuous functions

Consider the mapping given by

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x) := x^2.$$

Then, given any open interval $]a, b[$ (with $a < b$), we may compute the pre-image of this interval:



One gets:

$$f^{-1}(]a, b[) = \begin{cases} \varnothing & \text{, if } b \le 0 \\ ]-\sqrt{b}, \sqrt{b}[& \text{, if } a < 0 < b \\ ]-\sqrt{b}, \sqrt{b}[\setminus\{0\} & \text{, if } a = 0 < b \\ ]-\sqrt{b}, -\sqrt{a}[\cup]\sqrt{a}, \sqrt{b}[& \text{, if } 0 < a < b \end{cases}$$

and we may conclude:

$$\boxed{\text{the pre-image of an open interval is open.}}$$

But every open set O in $\mathbb{R}$ can be written as a finite or at most countable union of open intervals:

$$O = \bigcup_{\iota \in I}]a_\iota, b_\iota[\qquad \text{with } a_\iota < b_\iota$$

where $I = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or $I = \mathbb{N}$.

Indeed, if $\{q_n \ : \ n \in \mathbb{N}\}$ denotes all the rational numbers in O, then O can be written as a countable union of open intervals in the following way:

$$O = \bigcup_{\substack{n,m \in \mathbb{N}: q_n < q_m \\ ]q_n, q_m[\subset O}}]q_n, q_m[.$$

Thus we get

$$f^{-1}(O) = \underbrace{\bigcup_{\iota \in I} \underbrace{f^{-1}(]a_\iota, b_\iota[)}_{\text{open}}}_{\text{open}},$$

i.e.

$$\boxed{\text{the pre-image of an open set is open!}}$$

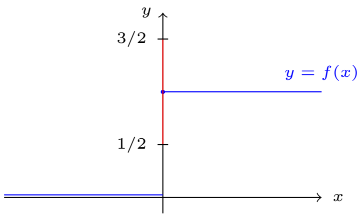
i.e.

$$\boxed{f^{-1}(\mathscr{O}^1) \subset \mathscr{O}^1}.$$

Pre-images under discontinuous functions

Consider the function

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x) := \begin{cases} 1 & \text{, if } x \geq 0 \\ 0 & \text{, if } x < 0. \end{cases}$$



Then

$$O :=]\frac{1}{2}, \frac{3}{2}[\quad \text{is an open set}$$

but

$$f^{-1}(O) = [0, +\infty[\quad \text{is } \textit{not} \text{ an open set.}$$

Thus

$$f^{-1}(\mathscr{O}^1) \not\subset \mathscr{O}^1.$$

A general result about the pre-image of open sets under a continuous function

Proposition 58.

Hyp *Consider a mapping*

$$f : X \to Y, \quad \text{where } X \text{ and } Y \text{ are topological spaces.}$$

We denote by $\mathscr{O}(X)$ *and* $\mathscr{O}(Y)$ *the families of open sets:*

$$\mathscr{O}(X) := \{O \subset X \ : \ O \text{ is open in } X\}$$

and

$$\mathscr{O}(Y) := \{O \subset Y \ : \ O \text{ is open in } Y\}$$

Concl *Then, the mapping* f *is continuous, i.e.*

$$x_n \to x \Longrightarrow f(x_n) \to f(x)$$

if and only if
the pre-image of an open set in Y *is an open set in* X*, i.e. if and only if*

$$\forall O \in \mathscr{O}(Y), \qquad f^{-1}(O) \in \mathscr{O}(X),$$

i.e. if and only if

$$f^{-1}(\mathscr{O}(Y)) \subset \mathscr{O}(X).$$

The notion of numeric functions

Definition 59.
Let X be a universe.
We call every mapping

$$f : X \to \overline{\mathbb{R}} = [-\infty, +\infty], \quad x \mapsto f(x)$$

a numeric function.

Example 60.
So

$$f : \mathbb{R} \to \overline{\mathbb{R}} = [-\infty, +\infty], x \mapsto f(x) := \begin{cases} 1/x & \text{if } x \neq 0 \\ +\infty & \text{if } x = 0 \end{cases}$$

is a numeric function.

Remark 61. *Every function* $f : X \to \mathbb{R} =]-\infty, +\infty[$, $x \mapsto f(x)$ *can be considered as a numeric function!*

The notion of measurable (numeric) function

Definition 62.
Let $(X, \mathscr{A})$ be a measurable space.

1. $\mathscr{A}$-measurable (numeric) function:

   A mapping $f : X \to \overline{\mathbb{R}}$ such that
   $$\forall \alpha \in \mathbb{R}, \qquad f^{-1}(]\alpha, +\infty]) := \{x \in \mathbb{R} \ : \ f(x) > \alpha\} \in \mathscr{A}$$

2. $\mathscr{A}$-measurable complex-valued function:

   A mapping $f : X \to \mathbb{C}$, $x \mapsto f(x) := (\Re f)(x) + i(\Im f)(x)$ such that the real and the imaginary part of f
   $$\Re f, \Im f : \mathbb{R} \to \mathbb{R}$$
   are both $\mathscr{A}$-measurable functions.

Continuous functions are λ^p-measurable

Proposition 63.

Hyp *Consider a continuous function $f : \mathbb{R}^p \to \mathbb{R}$ ($p = 1, 2, 3, \ldots$) defined on the measure space $(\mathbb{R}^p, \mathscr{B}(\mathbb{R}^p), \lambda^p)$.*

Concl *Then this continuous function f is $\mathscr{L}^p$-measurable.*

Proof. This follows from the fact that the pre-image of an open set by a continuous function is open:
$$f^{-1}(]\alpha, +\infty]) = f^{-1}(\underbrace{]\alpha, +\infty[}_{\text{open}}) \in \mathscr{O}^p \subset \mathscr{B}(\mathbb{R}^p).$$
□

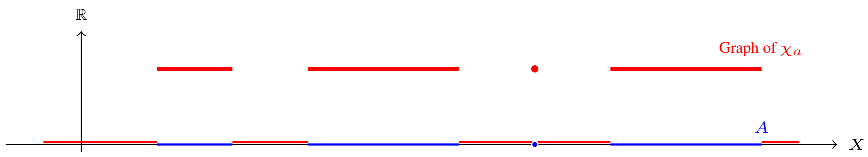
Measurable functions need not to be continuous

There exist many measurable functions that are *not* continuous. In order to show this in an easy way, we introduce a notation that will be useful in what follows:

Definition 64.

<u>Given:</u>	A subset A of a universe X
we define:	<u>the characteristic function χ_A</u> as:
	the function

$$\chi_A : X \to \mathbb{R}, \qquad x \mapsto \chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$



Proposition 65.
A characteristic function χ_A defined on a measurable space $(X, \mathscr{A})$ is $\mathscr{A}$-measurable if and only if $A \in \mathscr{A}$.

Proof. Indeed, for all $\alpha \in \mathbb{R}$, the pre-images $\chi_A^{-1}(]\alpha, +\infty])$ are one of the following subsets:

$$\underbrace{\text{the empty set } \varnothing}_{\in \mathscr{A}} \qquad \text{or} \qquad A \qquad \text{or} \qquad \underbrace{\text{the whole universe } X}_{\in \mathscr{A}}.$$

Thus, χ_A is $\mathscr{A}$-measurable if and only if $A \in \mathscr{A}$. □

Equivalent definitions for a function to be measurable

Proposition 66.

<u>*Hyp*</u> *Let $f : X \to \overline{\mathbb{R}}$ be a (numeric) function defined on a measurable space$(X, \mathscr{A})$.*

Concl *The 4 following conclusions are equivalent:*

1. *f is $\mathscr{A}$-measurable, i.e.*

$$\forall \alpha \in \mathbb{R}, \qquad f^{-1}(]\alpha, +\infty]) := \{x \in \mathbb{R} \ : \ f(x) > \alpha\} \in \mathscr{A}.$$

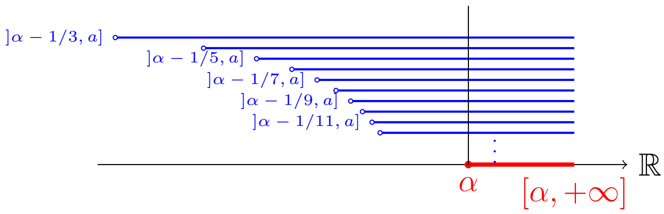
2. $\forall \alpha \in \mathbb{R}, \qquad f^{-1}([\alpha, +\infty]) := \{x \in \mathbb{R} \ : \ f(x) \geq \alpha\} \in \mathscr{A}.$

3. $\forall \alpha \in \mathbb{R}, \qquad f^{-1}([-\infty, \alpha[) := \{x \in \mathbb{R} \ : \ f(x) < \alpha\} \in \mathscr{A}.$

4. $\forall \alpha \in \mathbb{R}, \qquad f^{-1}(-\infty, \alpha]) := \{x \in \mathbb{R} \ : \ f(x) \leq \alpha\} \in \mathscr{A}.$

Proof. We only show that the first point implies the second point. This follows from

$$f^{-1}([\alpha, +\infty]) = f^{-1}(\bigcap_{n=1}^{\infty}]\alpha - 1/n, +\infty] = \underbrace{\bigcap_{n=1}^{\infty} \underbrace{f^{-1}(]\alpha - 1/n, +\infty]}_{\in \mathscr{A}}}_{\in \mathscr{A}}.$$



The other implications can be proven in a similar way! □

Properties of measurable functions

Proposition 67.

Hyp

- *$(X, \mathscr{A})$ be a measurable space and*
- *f and g be $\mathscr{A}$-measurable (numeric) functions defined on X.*

<u>*Concl*</u>

1. *For all* $\alpha \in \mathbb{R}$*, the (numeric) function*

$$\alpha f : X \to \overline{\mathbb{R}}, \quad x \mapsto (\alpha f)(x) := \alpha \cdot f(x)$$

(with our conventions for computing in $\overline{\mathbb{R}}$*) is* $\mathscr{A}$*-measurable, too.*

2. *If the sum* $f(x) + g(x)$ *is defined for all* $x \in X$ *(with our conventions for computing in* $\overline{\mathbb{R}}$*), the (numeric) function*

$$f + g : X \to \overline{\mathbb{R}}, \quad x \mapsto (f+g)(x) := f(x) + g(x)$$

is $\mathscr{A}$*-measurable, too.*

3. *The (numeric) function*

$$f \cdot g : X \to \overline{\mathbb{R}}, \quad x \mapsto (f \cdot g)(x) := f(x) \cdot g(x)$$

(with our conventions for computing in $\overline{\mathbb{R}}$*) is* $\mathscr{A}$*-measurable, too.*

4. *The (numeric) functions*

$$\max\{f, g\} : X \to \overline{\mathbb{R}}, \quad x \mapsto (\max\{f,g\})(x) := \max\{f(x), g(x)\}$$

and

$$\min\{f, g\} : X \to \overline{\mathbb{R}}, \quad x \mapsto (\min\{f,g\})(x) := \min\{f(x), g(x)\}$$

(with our conventions for computing in $\overline{\mathbb{R}}$*) is* $\mathscr{A}$*-measurable, too.*

Proof. If $\alpha \neq 0$ the first part follows from

$$\forall \gamma \in \mathbb{R}, \quad \{x \in X \ : \ \alpha \cdot f(x) > \gamma\} = \begin{cases} \underbrace{\{x \in X \ : \ f(x) > \gamma/\alpha\}}_{\in \mathscr{A}} & \text{, if } \alpha > 0 \\ \underbrace{\{x \in X \ : \ f(x) < \gamma/\alpha\}}_{\in \mathscr{A}} & \text{, if } \alpha < 0. \end{cases}$$

If $\alpha = 0$, the sets $\{x \in X \ : \ \alpha \cdot f(x) > \gamma\}$ are equal to either X or $\varnothing$, and both of these sets belongs to $\mathscr{A}$.

For the last part, the following argument can be used: $\forall \gamma \in \mathbb{R}$,

$$\{x \in X \ : \ \min\{f(x), g(x)\} < \gamma\} = \underbrace{\underbrace{\{x \in X \ : \ f(x) < \gamma\}}_{\in \mathscr{A}} \cup \underbrace{\{x \in X \ : \ g(x) < \gamma\}}_{\in \mathscr{A}}}_{\in \mathscr{A}}.$$

We do not give proofs for the other parts. □

The positive part and the negative part of measurable functions

Definition 68.

Let f be a (numeric) function defined on a measurable space $(X, \mathscr{A})$. Then we define:

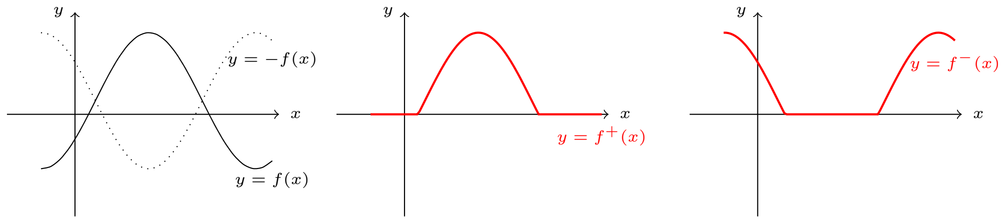
1. the positive part f^+ of f as:

$$f^+ : X \to \overline{\mathbb{R}}, \quad x \mapsto f^+(x) := \max\{0, f(x)\}.$$

2. the negative part f^- of f as:

$$f^- : X \to \overline{\mathbb{R}}, \quad x \mapsto f^-(x) := \max\{0, -f(x)\}.$$

Remark 69. *Be careful, both f^+ and f^- are non-negative functions!*



Proposition 70.

One has

$$f = f^+ - f^-$$

and

$$|f| = f^+ + f^-$$

Proposition 71.

<u>*Hyp*</u>

- $(X, \mathscr{A})$ *be a measurable space and*
- f *be an* $\mathscr{A}$*-measurable (numeric) function defined on* X.

<u>*Concl*</u>

1. *The positive part* f^+ *and the negative part* f^- *are* $\mathscr{A}$*-measurable;*
2. *The absolute value* $|f|$ *is* $\mathscr{A}$*-measurable.*

Measurability is preserved by limits

Proposition 72.

<u>*Hyp*</u> *Let*

- $(X, \mathscr{A})$ *be a measurable space and*
- $\{f_n\}_{n=1}^{+\infty}$ *a sequence of* $\mathscr{A}$*-measurable (numeric) functions defined on* X.

<u>*Concl*</u>

1. *The (numeric) functions* $f(x) := \inf_{n\in\mathbb{N}^*} f_n(x)$ *and* $g(x) := \sup_{n\in\mathbb{N}^*} f_n(x)$ *are both* $\mathscr{A}$*-measurable.*
2. *If the limit* $\lim_{n\to\infty} f_n(x)$ *exists (in* $\overline{\mathbb{R}}$*) for all* $x \in X$*, then the function*

$$\lim_{n\to\infty} f_n : X \to \overline{\mathbb{R}}, \quad x \mapsto \lim_{n\to\infty} f_n(x)$$

is $\mathscr{A}$*-measurable, too.*

Proof. Concerning the first part, we may argue as follows: $\forall \alpha \in \mathbb{R}$,

$$\{x \in X \ : \ \sup_{n\in\mathbb{N}^*} f_n(x) > \alpha\} = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{\{x \in X \ : \ f_n(x) > \alpha\}}_{\in\mathscr{A}}}_{\in\mathscr{A}}$$

and

$$\{x \in X \;:\; \inf_{n\in\mathbb{N}^*} f_n(x) < \alpha\} = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{\{x \in X \;:\; f_n(x) < \alpha\}}_{\in \mathscr{A}}}_{\in \mathscr{A}}$$

□

A class of simple functions

Let $(X, \mathscr{A})$ be a measurable space (so that $\mathscr{A}$ is a σ-algebra over X).
Suppose given, for some fixed $n \in \{1, 2, 3, \dots\}$,

- $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$, not necessarily all distinct and
- $A_1, A_2, \dots, A_n \in \mathscr{A}$, not necessarily pairwise distinct.

Then the function

$$f := \sum_{k=1}^{n} \alpha_k \cdot \chi_{A_k}$$

is $\mathscr{A}$-measurable. Remark that such a function has a finite range!

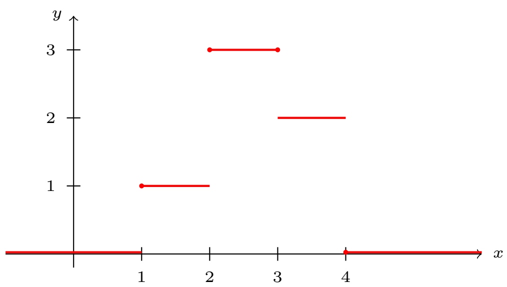
Example 73. Consider in $X = \mathbb{R}$ equipped with $\mathscr{A} = \mathscr{B}(\mathbb{R})$ the function

$$f(x) := \chi_{[1,3]}(x) + 2 \cdot \chi_{[2,4[}(x).$$

Then f has a finite range: $\{0, 1, 2, 3\}$.

We have

$$f = 0 \cdot \chi_{]-\infty,1[} + 1 \cdot \chi_{[1,2[} + 3 \cdot \chi_{[2,3]} + 2 \cdot \chi_{]3,4[} + 0 \cdot \chi_{[4,+\infty[}.$$



The concept of step-function

This motivates the following definition:

Definition 74.
Let $(X, \mathscr{A})$ be a measurable space.
A numeric function f is called a *step-function* or *simple function* if and only if

1. f is a $\mathbb{R}$-valued function, i.e. $f : X \to \mathbb{R} =]-\infty, +\infty[$;

2. f is $\mathscr{A}$-measurable and
3. the range of f is finite, i.e. of the form $\{\beta_1, \ldots, \beta_n\}$.

Remark 75. *Then f can be written as*

$$f(x) = \sum_{k=1}^{n} \beta_k \cdot \chi_{B_k}, \qquad \text{with } B_k := f^{-1}(\{\beta_k\}) \in \mathscr{A} \text{ (for k=1,2,...,n).}$$

Remark 76. *Every function of the form*

$$f := \sum_{k=1}^{n} \alpha_k \cdot \chi_{A_k}$$

with $A_k \in \mathscr{A}$ and $\alpha_k \in \mathbb{R}$ (for $k = 1, 2, \ldots, n$) is a step-function defined on the measurable space $(X, \mathscr{A})$.

The family of measurable functions

Let us introduce a notation:

Definition 77.
Let $(X, \mathscr{A})$ be a measurable space.

1. The family of measurable functions $\mathscr{L}(X, \mathscr{A})$):

$$\mathscr{L}(X, \mathscr{A}) := \{f : X \to \mathbb{R} \ : \ f \text{ is } \mathscr{A}, \mathscr{B}(\mathbb{R})\text{-measurable}\}.$$

2. The family of non-negative, measurable functions $\mathscr{L}^+(X, \mathscr{A})$):

$$\mathscr{L}^+(X, \mathscr{A}) := \{f \in \mathscr{L}(X, \mathscr{A}) \ : \ f(x) \geq 0, \quad \forall x \in X\}.$$

$\mathscr{L}(X, \mathscr{A})$ is function space. By this we mean that

$$\left.\begin{array}{l} f, g \in \mathscr{L}(X, \mathscr{A}) \\ \alpha \in \mathbb{R} \end{array}\right\} \Longrightarrow f + g, f \cdot g, \alpha f, |f| \in \mathscr{L}(X, \mathscr{A}).$$

The family of measurable numeric functions

Let us introduce a notation:

Definition 78.
Let $(X, \mathscr{A})$ be a measurable space.

1. The family of measurable, numeric functions $\overline{\mathscr{Z}}(X, \mathscr{A})$:

$$\overline{\mathscr{Z}}(X, \mathscr{A}) := \{f : X \to \overline{\mathbb{R}} \ : \ f \text{ is } \mathscr{A}, \mathscr{B}(\overline{\mathbb{R}})\text{-measurable}\} .$$

2. The family of non-negative, measurable, numeric functions $\overline{\mathscr{Z}}^+(X, \mathscr{A})$:

$$\overline{\mathscr{Z}}^+(X, \mathscr{A}) := \{f \in \overline{\mathscr{Z}}(X, \mathscr{A}) \ : \ f(x) \geq 0, \quad \forall x \in X\}.$$

$\overline{\mathscr{Z}}(X, \mathscr{A})$ is a function space. By this we mean that

$$\left.\begin{array}{l} f, g \in \overline{\mathscr{Z}}(X, \mathscr{A}) \\ \alpha \in \mathbb{R} \end{array}\right\} \Longrightarrow f + g, f \cdot g, \alpha f, |f| \in \overline{\mathscr{Z}}(X, \mathscr{A}).$$

The set of simple functions as a subspace

Definition 79.
Given: $(X, \mathscr{A})$ a measurable space (so that $\mathscr{A}$ is a σ-algebra over X).
we define: the set of step-functions (or the set of simple functions) as:

$$\mathscr{T}(X, \mathscr{A}) := \{f : X \to \overline{\mathbb{R}} \ : \ f \text{ a step-function (and hence } \mathscr{A}\text{-measurable)}\}$$

and we put

$$\mathscr{T}^+(X, \mathscr{A}) := \{f \in \mathscr{T}(X, \mathscr{A}) \ : \ f(x) \geq 0, \forall x \in X\} .$$

Remark 80. *It is easy to see that*

$$\left.\begin{array}{l} f, g \in \mathscr{T}(X, \mathscr{A}) \\ \alpha \in \mathbb{R} \end{array}\right\} \Longrightarrow f + g, f \cdot g, \alpha f, |f| \in \mathscr{T}(X, \mathscr{A}).$$

Thus, $\mathscr{T}(X, \mathscr{A})$ is a sub-space of the space of (numeric) functions.

The remarkable fact about the space $\mathscr{T}(X, \mathscr{A})$ is that, despite the simple structure of its members, this space is dense in the set of all measurable numeric functions.

This is the central message of the following proposition. Before we formulate this result, we recall that a numeric function $f : X \to \overline{\mathbb{R}}$ is

- *finite* if and only if

$$-\infty < f(x) < +\infty, \quad \forall x \in X, \text{ i.e. } \pm\infty \notin f(X)$$

and

- f is *bounded* if and only if there exists some constant $M \in]0, +\infty[$, such that

$$-M \leq f(x) \leq M, \quad \forall x \in X.$$

Approximation of measurable numeric functions by step functions

Proposition 81.

<u>*Hyp*</u>

- *$(X, \mathscr{A})$ be a measurable space (so that $\mathscr{A}$ is a σ-algebra over X), let*
- *$f \in \overline{\mathscr{L}}^{+}(X, A)$.*

<u>*Concl*</u> *There exists a sequence $\{u_n\}_{n=1}^{+\infty}$ of positive step-functions (i.e. $u_n \in \mathscr{T}^{+}(X, \mathscr{A})$) with*

$$u_n \nearrow f \qquad \text{as } n \to \infty$$

i.e.

- *the sequence $\{u_n\}_{n=1}^{+\infty}$ is non-decreasing, i.e. $u_n(x) \leq u_{n+1}(x)$, $\forall x \in X$ and $\forall n \in \{1, 2, 3, \ldots\}$;*
- *$u_n(x) \leq f(x)$, $\forall x \in X$ and $\forall n \in \{1, 2, 3, \ldots\}$;*
- $\lim_{n\to\infty} u_n(x) = f(x)$, $\forall x \in X$.

Moreover, if f is bounded, the convergence $u_n \nearrow f$ is uniform, *i.e.*

$$\begin{aligned}&\forall \text{ given tolerance } \varepsilon > 0\\ &\exists \text{ a threshold } n_0 \text{ (that depends on the given } \varepsilon) \text{ such that}\\ &\qquad |u_n(x) - f(x)| < \varepsilon, \quad \forall x \in X \text{ as soon as } n \geq n_0\end{aligned}$$

i.e.

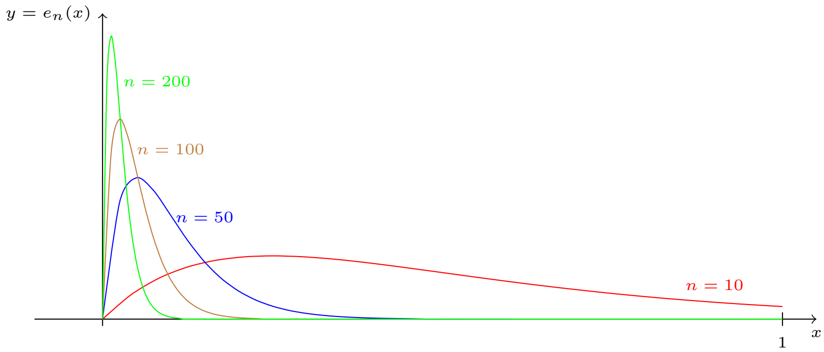
$$\begin{aligned}&\forall \text{ given tolerance } \varepsilon > 0\\ &\exists \text{ a threshold } n_0 \text{ (that depends on the given } \varepsilon) \text{ such that}\\ &\qquad \sup_{x \in X} |u_n(x) - f(x)| \leq \varepsilon, \quad \forall n \geq n_0.\end{aligned}$$

Before giving the proof, let us explore the concept of uniform convergence.

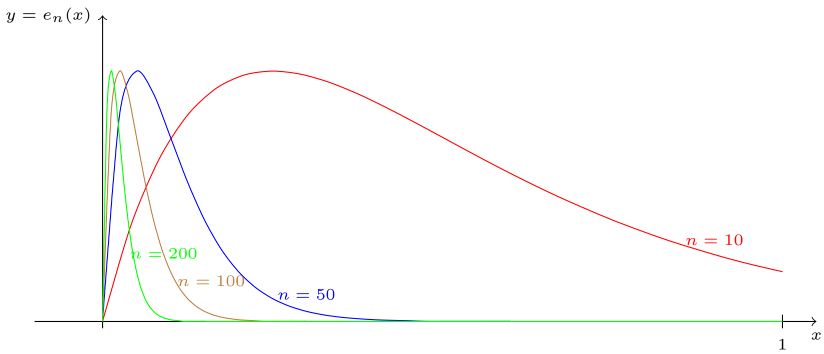
Let us suppose that $f : [0,1] \to \mathbb{R}$ and that $u_n \nearrow f$. Then we may consider the error functions

$$e_n(x) := f(x) - u_n(x).$$

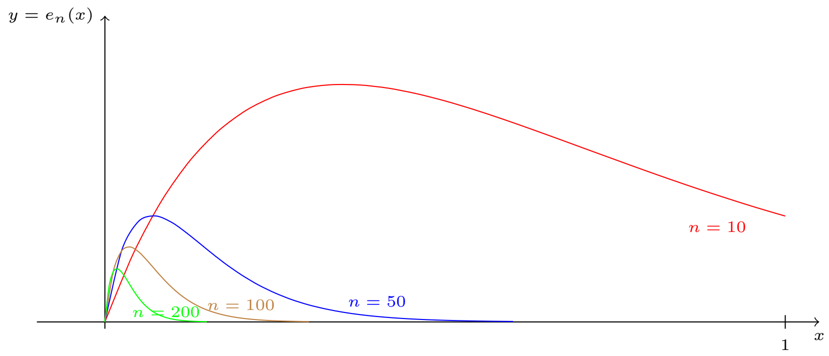
This is an illustration of a non-uniform convergence:



This is an illustration of a non-uniform convergence, too:



This is an illustration of a uniform convergence:



Proof. (of Proposition 81)

Put

$$u_n(x) := \begin{cases} \frac{k}{2^n} & \text{, if } \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \text{ for some } k \in \{0, 1, 2, \ldots, 2^{2n} - 1\} \\ 2^n & \text{, if } 2^n \leq f(x) \end{cases}$$

Then $\{u_n\}_{n=1}^{+\infty}$ is a non-decreasing sequence of positive step-functions.

- If $f(x) < +\infty$, there exists some N such that $f(x) < 2^n$ for $n > N$. Thus, for $n > N$, we have

$$0 \leq f(x) - u_n(x) \leq 1/2^n.$$

- If $f(x) = +\infty$, we have $u_n(x) = 2^n$

In both cases, we have $\lim_{n \to \infty} u_n(x) = f(x)$ for all $x \in \mathbb{R}$.

If f is bounded, this convergence is uniform since

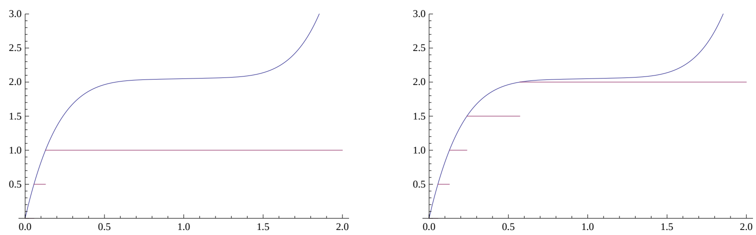
$$\sup_{x \in \mathbb{R}} |f(x) - u_n(x)| \leq 1/2^n \to 0 \qquad \text{as } n \to \infty.$$

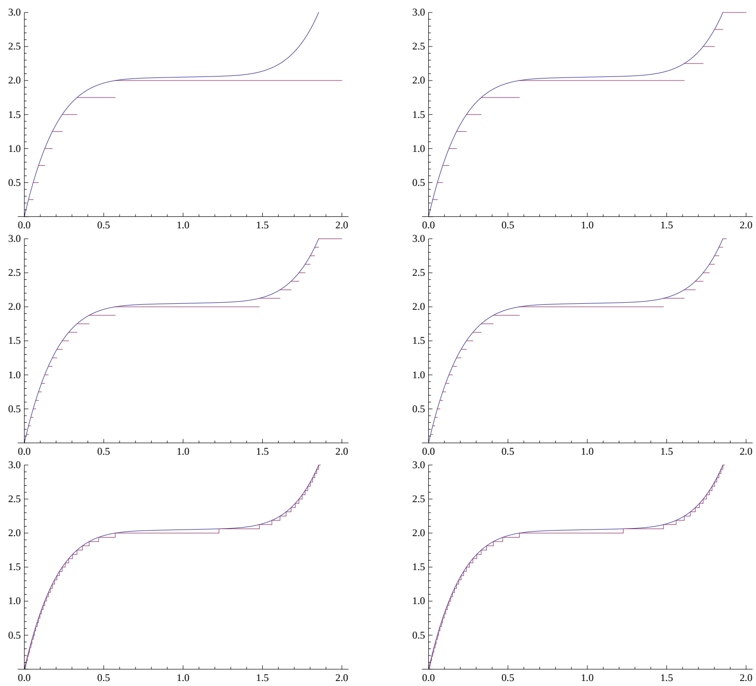
□

Illustration of the above proof

The above proof relies on induction, where each step consists of two parts:

1. increasing the range by 1 and
2. subdivide this range by a factor 2.





Extension to measurable numeric functions that are bounded from below

Proposition 82.

<u>Hyp</u> *Suppose that*

- $(X, \mathscr{A})$ *is a measurable space and that*
- $f \in \overline{\mathscr{Z}}(X, A)$.

<u>Concl</u> *If f is bounded from below, then*

$$\exists \{u_n\}_{n=1}^{+\infty} \textit{ in } \mathscr{T}(X, \mathscr{A}) \textit{ with } u_n \nearrow f.$$

Proof. There exists some $M \in \mathbb{R}$ with $f(x) \geq M, \forall x \in X$. Then $f(x) - M \in \overline{\mathscr{Z}}^+(X, \mathscr{A})$; thus

$$\exists \{v_n\}_{n=1}^{+\infty} \text{ in } \mathscr{T}^+(X, \mathscr{A}) \text{ with } v_n \nearrow f - M.$$

If we put $u_n := v_n + M$ we get a sequence of (not necessarily non-negative) step functions $\{u_n\}_{n=1}^{+\infty}$ with

$$u_n = \underbrace{v_n + M}_{\in \mathscr{T}(X, \mathscr{A})} \nearrow f.$$

□

The density of $\mathscr{T}(X,\mathscr{A})$ in $\overline{\mathscr{L}}(X,\mathscr{A})$

Any (numeric) function $f \in \overline{\mathscr{L}}(X,\mathscr{A})$ can be written as

$$f = f^+ - f^-, \qquad \text{, where } f^+, f^- \in \overline{\mathscr{L}}^+(X,\mathscr{A}).$$

Hence, there exist non-decreasing sequences

$$\{u_n\}_{n=1}^{+\infty} \text{ and } \{v_n\}_{n=1}^{+\infty} \qquad \text{in } \mathscr{T}^+(X,\mathscr{A})$$

with

$$u_n \nearrow f^+ \quad \text{and} \quad v_n \nearrow f^-.$$

Remark that $u_n - v_n \in \mathscr{T}(X,\mathscr{A})$ and that

$$\lim_{n\to\infty}(u_n - v_n) = f^+ - f^- = f.$$

Proposition 83.

Hyp $f \in \overline{\mathscr{L}}(X,\mathscr{A})$

Concl *Then there exists a sequence* $\{w_n\}_{n=1}^{+\infty}$ *of step functions with*

$$\lim_{n\to\infty} w_n = f.$$

Remark, however, that this sequence may be non-monotonous.

2

Integrating measurable functions

Our aim is to give a (new) definition of the integral in such a way that

1. the new definition $\int_X f \; d\mu$ contains (if possible) the old definition of the Riemannian Integral (R)- $\int f \; dx$;

2. we will be able to integrate some functions like

$$f(x) = \begin{cases} 1 & \text{, if } x \in \mathbb{Q} \\ 0 & \text{, if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

   whose Riemannian integral does not exist.

3. we have powerful theorems around exchanging limits and integrals like

$$\int_X \left(\lim_{n\to\infty} f_n \right) \; d\mu = \lim_{n\to\infty} \int_X f_n \; d\mu$$

   involving weak hypothesis around the kind of convergence for the sequence $\{f_n\}_{n=1}^{+\infty}$.

Remark 84. *For Riemannian integrals, hypothesis for point 3 above are rather strong like* 'uniform *convergence of the* continuous *functions f_n' if X is a bounded interval.*

Remark 85. *We will achieve point 1 above as long as $X = [a, b]$ is a finite interval, but we will fail for intervals including $+\infty$ and/or $-\infty$.*

Remark 86. *Points 2 and 3 are related as we can see it in the following example.*

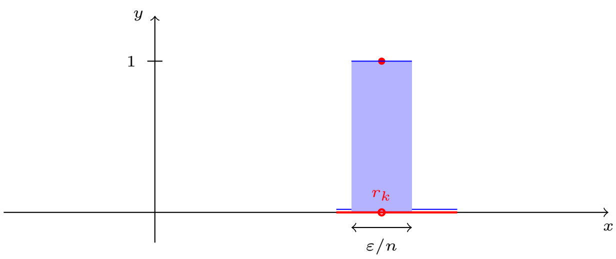
Example 87.
Consider the countable set

$$\mathbb{Q} \cap [0, 1] = \{r_1, r_2, r_3, \ldots\}$$

and the sequence of functions $\{f_n\}_{n=1}^{+\infty}$ given by

$$f_n : [0, 1] \to \mathbb{R}, \qquad x \mapsto f_n(x) = \begin{cases} 1 & \text{, for } x \in \{r_1, r_2, r_3, \ldots, r_n\} \\ 0 & \text{, elsewhere.} \end{cases}$$

Clearly, (R)- $\int_0^1 f_n(x) \; dx = 0$, since the lower sums are equal to 0, and since the upper sums can be made arbitrary small (remember, that the set $\{r_1, r_2, \ldots r_n\}$ is finite, so all points $r_1, r_2, \ldots, r_n$ are isolated).



So $\lim_{n\to\infty}$ (R)- $\int_0^1 f_n(x)\,dx = 0$. On the other hand,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{, if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{, elsewhere} \end{cases}$$

so (R)- $\int_0^1 (\lim_{n\to\infty} f_n(x))\,dx$ does not exist: the lower sums are equal to 0, whereas the upper sums all equal 1.

Hence, we *cannot* have

$$\boxed{\text{(R)-} \int_0^1 (\lim_{n\to\infty} f_n(x))\,dx = \lim_{n\to\infty} \text{(R)-} \int_0^1 f_n(x)\,dx.}$$

2.1. Integration of step functions

Integration of a measurable characteristic function

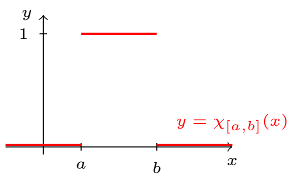
Let us consider a measure space $(X, \mathscr{A}, \mu)$, where $\mathscr{A}$ is a σ-algebra over X and where μ is a measure defined on $\mathscr{A}$.

In order to define the integral

$$\int_X \chi_A\,d\mu, \qquad \text{where } A \in \mathscr{A},$$

we look at the special case where

$$X = \mathbb{R}, \quad A = [a,b], \quad \mu = \lambda^1 \text{ (Lebesgue measure)}$$



(R)- $\int_{-\infty}^{+\infty} \chi_A\,dx = b - a = \lambda^1([a,b])$

Thus we put

$$\boxed{\int_X \chi_A(x)\,d\mu(x) = \int_X \chi_A\,d\mu = \mu(A), \qquad \forall A \in \mathscr{A}.}$$

Moreover, still by analogy with what happens for the 'Riemannian' case, we put

$$\boxed{\int_X \alpha \cdot \chi_A(x)\,d\mu(x) = \int_X \alpha\chi_A\,d\mu = \alpha\mu(A), \qquad \forall A \in \mathscr{A}, \forall \alpha \in \mathbb{R}}$$

i.e.

$$\boxed{\int_X \alpha \cdot \chi_A(x)\,d\mu(x) = \alpha \cdot \int_X \chi_A(x)\,d\mu(x)} \qquad \text{or in short} \qquad \boxed{\int_X \alpha\chi_A\,d\mu = \alpha \cdot \int_X \chi_A\,d\mu}$$

The integral of a step function: definition imposed by linearity

It would be nice, if our 'new' integral would be linear, in analogy to the 'Riemannian' case, again!

Thus we put

$$\begin{aligned} \int_X (\alpha \cdot \chi_A(x) + \beta \cdot \chi_B(x))\ d\mu(x) &= \alpha \int_X \chi_A\ d\mu + \beta \int_X \chi_B\ d\mu \\ &= \alpha\mu(A) + \beta\mu(B) \\ &\qquad \forall A, B \in \mathscr{A}, \quad \forall \alpha, \beta \in \mathbb{R}. \end{aligned}$$

But this rises the question whether or not the sum

$$\alpha\mu(A) + \beta\mu(B)$$

depends of the chosen form

$$f(x) = \alpha \cdot \chi_A(x) + \beta \cdot \chi_B(x)$$

for the function f we integrate.

So we are confronted to the following question:

$$\begin{aligned} &\alpha_1 \cdot \chi_{A_1}(x) + \alpha_2 \cdot \chi_{A_2}(x) \equiv \beta_1 \cdot \chi_{B_1}(x) + \beta_2 \cdot \chi_{B_2}(x) \Longrightarrow \\ &\qquad \Longrightarrow \alpha_1\mu(A_1) + \alpha_2\mu(A_2) = \beta_1\mu(B_1) + \beta_2\mu(B_2) \quad ? \end{aligned}$$

The integral of a step function: validation of the definition imposed by linearity

Fortunately, the result $\alpha\mu(A) + \beta\mu(B)$ does not depend on the representation chosen for the function $\alpha\chi_A + \beta\chi_B$.

Example 88.

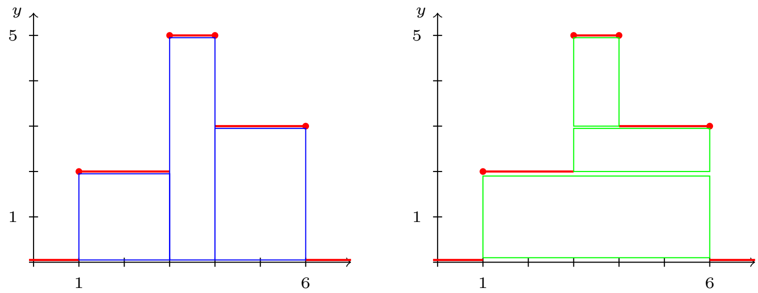
In $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda^1)$, we consider a step function f with two different representations:

$$\begin{aligned} f(x) &= 2 \cdot \chi_{[1,6]}(x) + \chi_{[3,6]}(x) + 2 \cdot \chi_{[3,4]}(x) \\ &= 2 \cdot \chi_{[1,3[}(x) + 5 \cdot \chi_{[3,4]}(x) + 3 \cdot \chi_{]4,6]}(x). \end{aligned}$$

Then $\int_{\mathbb{R}} f\ d\lambda^1$ does not depend on the chosen representation:

$$\begin{aligned} 2 \cdot \underbrace{\lambda^1([1,6])}_{=5} + 1 \cdot \underbrace{\lambda^1([3,6])}_{=3} + 2 \cdot \underbrace{\lambda^1([3,4])}_{=1} &= 15 \\ 2 \cdot \underbrace{\lambda^1([1,3[)}_{=2} + 5 \cdot \underbrace{\lambda^1([3,4])}_{=1} + 3 \cdot \underbrace{\lambda^1(]4,6])}_{=2} &= 15. \end{aligned}$$

The following picture gives a deeper insight into the above example:



This is a general result:

Proposition 89.

<u>*Hyp*</u> *Suppose that on the measure space* $(X, \mathscr{A}, \mu)$ *the step function*

$$f = \sum_{k=1}^{n} \alpha_k \cdot \chi_{A_k} \in \mathscr{T}(X, \mathscr{A})$$

with $A_k \in \mathscr{A}$ *for* $k = 1, 2, \dots, n$ *has another representation*

$$f = \sum_{j=1}^{m} \beta_j \cdot \chi_{B_j}, \qquad \text{where } B_j \in \mathscr{A} \text{ for } j = 1, 2, \dots, m.$$

<u>*Concl*</u>

$$\sum_{k=1}^{n} \alpha_k \mu(A_k) = \sum_{j=1}^{m} \beta_j \mu(B_j).$$

The integral of a step function: final version

Thus the following definition makes sense:

Definition 90.

Given: the step function

$$f = \sum_{k=1}^{n} \alpha_k \cdot \chi_{A_k} \in \mathscr{T}(X, \mathscr{A})$$

(with $A_k \in \mathscr{A}$, $\alpha_k \in \mathbb{R}$), where $(X, \mathscr{A}, \mu)$ is a measure space.

we define: the μ-integral of f over X (in the sense of Lebesgue) as:

$$\boxed{\int_X f \, d\mu = \int_X f(x) \, d\mu(x) = \sum_{k=1}^{n} \alpha_k \mu(A_k).}$$

Integrals and expectation value

Example 91.

Consider the space

$$X = \{\text{⚀}, \text{⚁}, \text{⚂}, \text{⚃}, \text{⚄}, \text{⚅}\} \text{ with } \mathscr{A} = \mathscr{P}(X) \text{ as } \sigma\text{-algebra},$$

and a probability

$$\mathbb{P} : \mathscr{P}(X) \to [0,1], \qquad \mathbb{P}(A) := \frac{|A|}{6}$$

Thus, for example,

$$\mathbb{P}(\{\text{⚂}, \text{⚀}\}) = \frac{2}{6} = \frac{1}{3}.$$

We introduce now a random variable $f \in \mathscr{T}(X, \mathscr{A})$ called 'number of points' via

$$\begin{aligned} f(x) \quad := \quad & 1 \cdot \chi_{\text{⚀}}(x) + 2 \cdot \chi_{\text{⚁}}(x) + 3 \cdot \chi_{\text{⚂}}(x) + \\ & +4 \cdot \chi_{\text{⚃}}(x) + 5 \cdot \chi_{\text{⚄}}(x) + 6 \cdot \chi_{\text{⚅}}(x). \end{aligned}$$

Then

$$\begin{aligned} \int_X f \, d\mathbb{P} \quad &= \quad 1 \cdot \mu(\text{⚀}) + 2 \cdot \mu(\text{⚁}) + 3 \cdot \mu(\text{⚂}) + \\ & \qquad 4 \cdot \mu(\text{⚃}) + 5 \cdot \mu(\text{⚄}) + 6 \cdot \mu(\text{⚅}) \\ &= \quad \frac{1+2+3+4+5+6}{6} = \frac{7}{2}. \end{aligned}$$

Thus we see that

$$\boxed{\int_X f \, d\mathbb{P} = \textit{expectation value } \mathbb{E}(f) \textit{ of } f}$$

$$\begin{array}{ccc} X = \{⚀,⚁,⚂,⚃,⚄,⚅\} & \xrightarrow{\quad f \quad} & \mathbb{R} \\ \mathscr{A} = \mathscr{P}(x) & & \\ \Big\downarrow \mathbb{P} & & \Big\downarrow \\ [0,1] & & \int_X f \, d\mathbb{P} \end{array}$$

Fundamental properties of the integral

The fundamental properties of an integral are

- the right gauge: $\int_X \chi_A \, d\mu = \mu(A)$,
- linearity and
- monotonicity.

The integral just defined is a mapping

$$\int_X \cdot \, d\mu : \mathscr{T}(X, \mathscr{A}) \to \mathbb{R}, \qquad f \mapsto \int_X f \, d\mu$$

exhibiting these three fundamental properties:

Proposition 92.

- **gauge:** $\int_X \chi_A \, d\mu = \mu(A)$, $\forall A \in \mathscr{A}$.
- **linearity:** $\forall \alpha \in \mathbb{R}$, $\forall f, g \in \mathscr{T}(X, \mathscr{A})$,

$$\int_X (\alpha f(x) + g(x)) \, d\mu(x) = \alpha \int_X f(x) \, d\mu(x) + \int_X g(x) \, d\mu(x).$$

- **Monotonicity:** $\forall f, g \in \mathscr{T}(X, \mathscr{A})$ *with* $f(x) \leq g(x)$, $\forall x \in X$,

$$\int_X f(x) \, d\mu(x) \leq \int_X g(x) \, d\mu(x),$$

i.e. inequalities can be integrated.

Proof. The only point that needs a proof is the last point. We have

$$\begin{aligned}\int_X g(x)\ d\mu(x) &= \int_X \Bigg[f(x) + \underbrace{(g-f)(x)}_{\in \mathscr{T}^+(X,\mathscr{A})} \Bigg]\ d\mu(x) \\ &= \int_X f(x)\ d\mu(x) + \underbrace{\int_X [g(x) - f(x)]\ d\mu(x)}_{\geq 0} \\ &\geq \int_X f(x)\ d\mu(x).\end{aligned}$$

□

2.2. The integral of positive, measurable numeric functions

Integral of a positive, measurable numeric function: definition imposed by monotonicity

Let us consider a positive, numeric function $f \in \overline{\mathscr{L}}^+(X,\mathscr{A})$, and let us try to define

$$\int_X f\ d\mu.$$

If we use a monotonicity argument, we can proceed as follows:

- choose a non-decreasing sequence $\{u_n\}_{n=1}^{+\infty}$ in $\mathscr{T}^+(X,\mathscr{A})$ with $u_n \nearrow f$; such a sequence exists by approximation (see above)!
- Put

$$\boxed{\int_X f\ d\mu = \int_X f(x)\ d\mu(x) = \lim_{n\to\infty} \underbrace{\int_X u_n\ d\mu}_{\text{integral of a step function}}.}$$

Integral of a positive, measurable numeric function: an important question about the above definition

However, there remains an open question:
Is the limit

$$\lim_{n\to\infty} \int_X u_n\ d\mu$$

the same for all possible choices of approximating sequences $\{u_n\}_{n=1}^{+\infty}$ in $\mathscr{T}^+(X,\mathscr{A})$ with $u_n \nearrow f$?

It can be shown that the answer is YES:

Proposition 93.
The limit

$$\lim_{n\to\infty}\int_X u_n\,d\mu$$

does not depend on the specific choice of the approximating sequences $\{u_n\}_{n=1}^{+\infty}$ *in the family* $\mathscr{T}^+(X,\mathscr{A})$ *with* $u_n \nearrow f$.

Integral of a positive, measurable numeric function: the final definition

Definition 94.
For any $f \in \overline{\mathscr{Z}^+}(X,\mathscr{A})$ we put

$$\boxed{\int_X f\,d\mu = \int_X f(x)\,d\mu(x) := \lim_{n\to\infty}\int_X u_n(x)\,d\mu(x)}$$

where $\{u_n\}_{n=1}^{+\infty}$ is any non-decreasing sequence in $\mathscr{T}^+(X,\mathscr{A})$ with

$$u_n \nearrow f \qquad (\text{as } n\to\infty)$$

and where the integrals

$$\int_X u_n(x)\,d\mu(x)$$

are defined as integrals of simple functions.

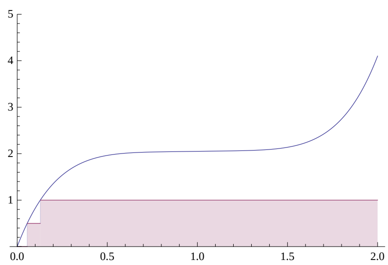
Remark 95. *It is important to insist on the fact that the above limit*

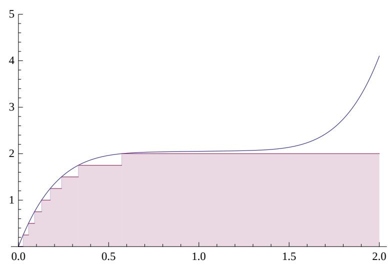
$$\lim_{n\to\infty}\int_X u_n(x)\,d\mu(x) \in [0,+\infty].$$

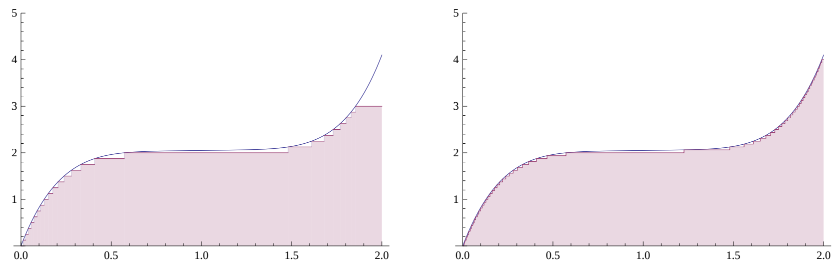
Thus, the value of the integral may take the value $+\infty$*!*

The following sequence of figures illustrate the above limit process. Due to the ‘horizontal’ cuts, the convergence is quick.

From a numerical point of view however, such an approximation process could be expensive!!







Example 96.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{, if } x \in \mathbb{Q} \\ 0 & \text{, elsewhere} \end{cases}$$

defined on the measure space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda^1)$.

Let $\mathbb{Q} = \{r_1, r_2, \ldots\}$ and consider the non-decreasing sequence $\{u_n\}_{n=1}^{+\infty}$ in $\mathscr{T}^+(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ given by

$$u_n(x) = \begin{cases} 1 & \text{, for } x \in \{r_1, r_2, \ldots, r_n\} \\ 0 & \text{, elsewhere.} \end{cases}$$

Then

- $u_n \nearrow f$ (as $n \to \infty$) and
- $$\int_{\mathbb{R}} u_n(x)\, d\lambda^1(x) = \int_{\mathbb{R}} \chi_{\{r_1,\ldots,r_n\}}(x)\ \lambda^1(x) = \sum_{k=1}^{n} 1 \cdot \lambda^1(\{r_k\}) = 0.$$

Thus

$$\boxed{\int_{\mathbb{R}} \chi_{\mathbb{Q}}(x)\, d\lambda^1(x) = 0.}$$

Remark that the above integral does not exist as a Riemannian integral!

But we can do even more!

Consider the numeric function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = +\infty \cdot \chi_{\mathbb{Q}}(x) = \begin{cases} +\infty & \text{, if } x \in \mathbb{Q} \\ 0 & \text{, elsewhere.} \end{cases}$$

Consider the non-decreasing sequence $\{u_n\}_{n=1}^{+\infty}$ in $\mathscr{T}^+(\mathbb{R}, \mathscr{B}(\mathbb{R})$ given by

$$u_n(x) = \begin{cases} n & \text{, for } x \in \{r_1, r_2, \ldots, r_n\} \\ 0 & \text{, elsewhere.} \end{cases}$$

Then

- $u_n \nearrow f$ (as $n \to \infty$) and
-
$$\int_{\mathbb{R}} u_n(x)\ d\lambda^1(x) = \sum_{k=1}^{n} n \cdot \lambda^1(\{r_k\}) = 0.$$

Thus
$$\boxed{\int_{\mathbb{R}} +\infty \cdot \chi_{\mathbb{Q}}(x)\ d\lambda^1(x) = 0.}$$

Remark that
$$\int_{\mathbb{R}} +\infty \cdot \chi_{\mathbb{Q}}(x)\ d\lambda^1(x) = +\infty \cdot \int_{\mathbb{R}} \chi_{\mathbb{Q}}(x)\ d\lambda^1(x)$$
at least in the present case: this is some *reinforced homogeneity*.

Integral of a positive, measurable numeric function: the fundamental properties

What we have got is the integral as a mapping
$$\int_X \cdot\ d\mu : \overline{\mathscr{Z}}^+(X, \mathscr{A}) \to [0, +\infty]$$
exhibiting the following version of modified fundamental properties:

- the right gauge: $\int_X \chi_A\ d\mu = \mu(A)$,
- linearity *as long as the scalars belong to* $[0, +\infty]$ and
- monotonicity.

With more details we have

Proposition 97.

- **gauge:** $\int_X \chi_A\ d\mu = \mu(A)$, $\forall A \in \mathscr{A}$.
- **additivity and strong homogeneity:** $\forall \alpha \in [0, +\infty]$, $\forall f, g \in \overline{\mathscr{Z}}^+(X, \mathscr{A})$,
$$\int_X (f(x) + g(x))\ d\mu(x) = \int_X f(x)\ d\mu(x) + \int_X g(x)\ d\mu(x).$$
and
$$\int_X \alpha \cdot f\ d\mu = \alpha \cdot \int_X f\ d\mu.$$

- **Monotonicity:** $\forall f, g \in \overline{\mathscr{L}}^+(X, \mathscr{A})$ *with* $f(x) \leq g(x)$, $\forall x \in X$,

$$\int_X f(x)\ d\mu(x) \leq \int_X g(x)\ d\mu(x).$$

Remark 98. *In the above proposition, one must use our conventions*

$$(+\infty) + (+\infty) = +\infty, \qquad 0 \cdot (+\infty) = 0, \qquad \ldots.$$

A final result

Proposition 99.
For any $f \in \overline{\mathscr{L}}^+(X, \mathscr{A})$ *we have*

$$\int_X f\ d\mu = 0 \Longleftrightarrow \mu\left(\{f > 0\}\right) = 0 \text{ i.e. } \{f > 0\} \text{ is a } \mu\text{-null-set.}$$

Remark 100. *Be careful, the hypothesis that* f *is non-negative cannot be dropped!*

We will only prove that

$$\mu\left(\{f > 0\}\right) > 0 \Longrightarrow \int_X f\ d\mu > 0.$$

The proof relies on the measurable sets

$$A := \{f > 0\} \qquad \text{and} \qquad A_n := \left\{f > \frac{1}{n}\right\} \quad (n \in \{1, 2, 3, \ldots\})$$

where $A_n \nearrow A$ and $\lim_{n\to\infty} \mu(A_n) = \mu(A) > 0$.

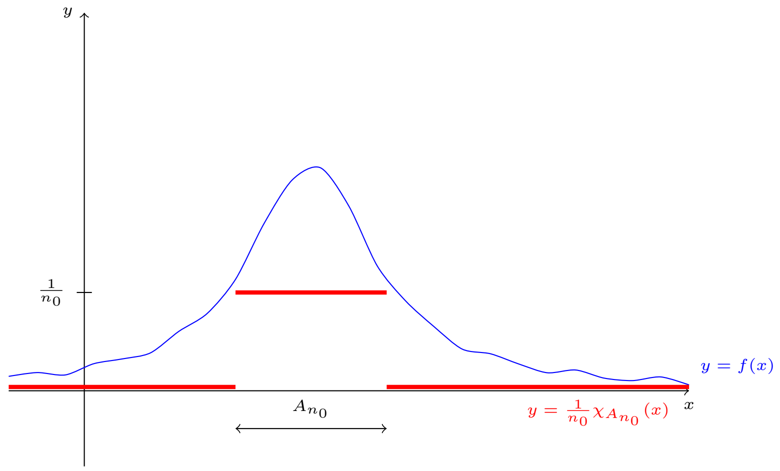
Proof. $\lim_{n\to\infty} \mu(A_n) = \mu(A) > 0$ implies that there exists some n_0 such that

$$\mu(A_{n_0}) > 0.$$

Since $f \geq \frac{1}{n_0} \cdot \chi_{A_{n_0}}$, we get by monotonicity of the integral

$$\int_X f\ d\mu \geq \int_X \frac{1}{n_0} \cdot \chi_{A_{n_0}}\ d\mu = \frac{1}{n_0} \cdot \mu(A_{n_0}) > 0.$$

□



2.3. The integrals of measurable numeric functions

Integrals and quasi-integrals

Recall that any $f \in \overline{\mathscr{L}}(X, \mathscr{A})$ can be written as

$$f = f^+ - f^-, \qquad \text{where } f^+, f^- \in \overline{\mathscr{L}}^+(X, \mathscr{A}).$$

Thus the following definitions make sense:

Definition 101.

Given: the numeric function $f \in \overline{\mathscr{L}}(X, \mathscr{A})$

we say: the function f is μ-integrable over X iff:

$$\int_X f^+ \, d\mu < +\infty \qquad \text{and} \qquad \int_X f^- \, d\mu < +\infty.$$

We put then

$$\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu$$

and we remark that $\int_X f \, d\mu \in \mathbb{R}$.
This integral is called the Lebesgue integral of f over X.

Definition 102.

<u>Given:</u> the numeric function $f \in \overline{\mathscr{L}}(X, \mathscr{A})$

we say: <u>f is μ-quasi-integrable over X</u> iff:

$$\int_X f^+ \, d\mu < +\infty \qquad \text{or} \qquad \int_X f^- \, d\mu < +\infty.$$

We put then

$$\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

Remark that if the result $\int_X f \, d\mu$ gives$+\infty$ or $-\infty$, the function f is *not* μ-integrable but only μ-quasi-integrable over X!

Remark 103. *Remark that any numeric function $f \in \overline{\mathscr{L}}^+(X, \mathscr{A})$ is μ-quasi-integrable, since in this case*

$$\int_X f^+ \, d\mu = \int_X f \, d\mu \in [0, +\infty] \qquad \textit{and} \qquad \int_X f^- \, d\mu = 0.$$

Other formulations for integrability

Proposition 104.

For any $f \in \overline{\mathscr{L}}(X, \mathscr{A})$, the following three statements are equivalent:

1. *f is μ-integrable over X;*
2. *f^+ and f^- are both μ-integrable over X;*
3. *$|f|$ is μ-integrable over X.*

Proof.

- point 1 $\Longleftrightarrow$ point 2: clear!
- point 2 $\Longrightarrow$ point 3: follows from $|f| = f^+ + f^-$.
- point 3 $\Longrightarrow$ point 2: follows from $f^+, f^- \leq |f|$.

$\square$

Definition 105.
Given: A measurable space $(X, \mathscr{A}, \mu)$
we define: The space of integrable (numeric) functions as:

$$\mathscr{L}^1(X, \mathscr{A}, \mu) := \left\{ f \in \overline{\mathscr{Z}}(X, \mathscr{A}) \ : \ \int_X |f(x)| \, d\mu(x) < +\infty \right\}$$

Integrable numeric functions are almost everywhere finite

Proposition 106.
For a given $f \in \mathscr{L}^1(X, \mathscr{A}, \mu)$ we have

$$\mu(\underbrace{\{x \in X \ : \ f(x) = \pm\infty\}}_{\{|f|=+\infty\}}) = 0,$$

i.e. $\{|f| = +\infty\}$ is a null-set.

Proof. Put

$$A := \{x \in X \ : \ |f(x)| = +\infty\}.$$

Then

$$+\infty \cdot \chi_A(x) \leq |f(x)|, \qquad \forall x \in X$$

so

$$+\infty \cdot \mu(A) \leq \int_X |f(x)| \, d\mu(x) < +\infty.$$

Thus $\mu(A) = 0$. □

Integral of measurable numeric functions: the fundamental properties

The integral as a mapping

$$\int_X \cdot \, d\mu : \mathscr{L}^1(X, \mathscr{A}, \mu) \to \mathbb{R}$$

has the following fundamental properties:

- the right gauge: $\int_X \chi_A \, d\mu = \mu(A)$,
- linearity and
- monotonicity.

With more details we have

Proposition 107.

- **gauge:** $\int_X \chi_A \, d\mu = \mu(A)$, $\forall A \in \mathscr{A}$.
- **linearity:** $\forall \alpha \in \mathbb{R}$, $\forall f, g \in \mathscr{L}^1(X, \mathscr{A}, \mu)$,

$$\int_X (\alpha f(x) + g(x)) \, d\mu(x) = \alpha \cdot \int_X f(x) \, d\mu(x) + \int_X g(x) \, d\mu(x).$$

- **Monotonicity:** $\forall f, g \in \mathscr{L}^1(X, \mathscr{A}, \mu)$ *with* $f(x) \leq g(x)$, $\forall x \in X$,

$$\int_X f(x) \, d\mu(x) \leq \int_X g(x) \, d\mu(x).$$

Remark 108. *In the above proposition, one must use our conventions*

$$(+\infty) + (+\infty) = +\infty, \qquad 0 \cdot (+\infty) = 0, \qquad \ldots .$$

Remark 109. *The proof of the second point is not straight forward, since*

$$(f+g)^+ \qquad \text{is not necessarily equal to} \qquad f^+ + g^+.$$

2.4. Integrals over measurable sets

Definition 110.
Given:

- a measure space $(X, \mathscr{A}, \mu)$ and a *measurable* subset $Y \subset X$
- a μ-integrable numeric function

$$f : X \to \overline{R}.$$

we define: the integral of f over Y as:

$$\int_Y f(x) \, d\mu(x) = \int_X \chi_Y(x) \cdot f(x) \, d\mu(x).$$

Remark 111. *If the function f is not given all over X, on can extend this function (for example by 0). The key property we need is that,*

$$\chi_Y \cdot f \in \overline{\mathscr{Z}}(X, \mathscr{A}).$$

Lebesgue-Stieltjes integrals over intervals

Example 112.

Consider now the special case where $X = \mathbb{R}$ and $\mathscr{A} = \mathscr{B}(\mathbb{R})$.

Suppose that the Lebesgue-Stieljes measure μ is given by a right-continuous, non-decreasing function g, that is continuous at a and b (with $a < b$); thus $\mu(\{a\}) = \mu(\{b\}) = 0$.

Then we put

$$\int_a^b f(x)\ d\mu(x) := \int_{[a,b]} f(x)\ d\mu(x) \left(= \int_{]a,b]} f(x)\ d\mu(x) = \cdots \right)$$

and

$$\int_b^a f(x)\ d\mu(x) := -\int_a^b f(x)\ d\mu(x).$$

Lebesgue integrals over intervals

Example 113.

If, in the above example, the measure μ is the Lebesgue measure λ^1, then

$$\int_a^b f(x)\ d\lambda^1(x) := \int_{[a,b]} f(x)\ d\lambda^1(x) \left(= \int_{]a,b]} f(x)\ d\lambda^1(x) = \cdots \right)$$

and

$$\int_b^a f(x)\ d\lambda^1(x) := -\int_a^b f(x)\ d\lambda^1(x).$$

'Almost nowhere change' result in preservation of the integral

It turns out that the Lebesgue integral $\int_X f\ d\mu$ is insensitive to changes made on the integrated function f as long as

- these changes preserve the measurability of the function and as long as
- these changes occur on a 'small' set of points x.

As a typical example, consider the following case:

Example 114.

For

$$f(x) = \begin{cases} 1 & \text{, if } x \in \mathbb{Q} \\ 0 & \text{, elsewhere} \end{cases}$$

we have $\int_{\mathbb{R}} f(x)\ d\lambda^1(x) = 0$.

If we modify this function f on the 'small' set $\mathbb{Q}$ in such a way that we obtain the function

$$g(x) = 0,$$

we have

$$\int_{\mathbb{R}} f(x)\, d\lambda^1(x) = \int_{\mathbb{R}} g(x)\, d\lambda^1(x) = 0.$$

Almost everywhere inequalities can be integrated over any subset...

Proposition 115.

Hyp $f, g \in \mathscr{L}^1(X, \mathscr{A}, \mu)$ *are such that*

$$f(x) \leq g(x) \qquad \mu\text{-a.e.}$$

Concl *For any* $A \in \mathscr{A}$, *we have*

$$\int_A f(x)\, d\mu(x) \leq \int_A g(x)\, d\mu(x).$$

Almost everywhere equalities can be integrated over any subset...

Proposition 116.

Hyp $f, g \in \mathscr{L}^1(X, \mathscr{A}, \mu)$ *are such that*

$$f(x) = g(x) \qquad \mu\text{-a.e.}$$

Concl *For any* $A \in \mathscr{A}$, *we have*

$$\int_A f(x)\, d\mu(x) = \int_A g(x)\, d\mu(x).$$

2.5. Series as integrals

Throughout this section, we choose

- $X = \{0, 1, 2, 3, \ldots\} =: \mathbb{N}_0$;
- $\mathscr{A} := \mathscr{P}(\mathbb{N}_0)$ and
- $\mu(A) = |A|$ (number of elements).

Then

$$(\mathbb{N}_0, \mathscr{P}(\mathbb{N}_0), \mu)$$

is a measure space.

What is a numeric function in this case?

A numeric function is a mapping

$$f : \mathbb{N}_0 \to \mathbb{R}, n \mapsto f_n.$$

Thus, a numeric function is a sequence $f_0, f_1, f_2, \ldots$ (where values of $\pm\infty$ are allowed!).

Remark that (numeric) functions are, in the present case, all measurable: this is due to the fact that the σ-algebra contains all subsets of $\mathbb{N}_0$.

How can a non-negative numeric function (i.e. a "sequence") be approximated by a simple function?

So let us consider a non-negative (numeric) function

$$f : \mathbb{N}_0 \to \mathbb{R}, n \mapsto f_n \qquad \text{with } f_n \geq 0.$$

If we set, for $m \in \{1, 2, 3, \ldots\}$,

$$g^{(m)} : \mathbb{N}_0 \to \mathbb{R}, n \mapsto g_n^{(m)} := \begin{cases} \min\{f_n, n\} & \text{, if } n \leq m \\ 0 & \text{, elsewhere} \end{cases}$$

then, the sequence of simple functions $g^{(m)}$ is non-decreasing with

$$g^{(m)} \nearrow f.$$

What is the integral of a non-negative function (i.e. "sequence")?

Now

$$\int_{\mathbb{N}_0} g_n^{(m)} \, d\mu(n) = \sum_{n=0}^{m} g_n^{(m)} \cdot \mu(\{n\}) = \sum_{n=0}^{m} g_n^{(m)} = \sum_{n=0}^{m} \min\{f_n, n\}.$$

Taking the limit $m \to \infty$, we obtain

$$\int_{\mathbb{N}_0} f_n \, d\mu(n) = \sum_{n=0}^{+\infty} f_n = \begin{cases} +\infty & \text{, if } \exists n \in \mathbb{N}_0 \text{ with } f_n = +\infty \\ \lim_{m\to\infty} \sum_{n=0}^{m} f_n & \text{, else.} \end{cases}$$

The meaning of integrable

Thus we get

Proposition 117.
A (numeric) function

$$f : \mathbb{N}_0 \to \mathbb{R}, n \mapsto f_n.$$

is integrable if and only if

1. *this function takes only real values (i.e. this mapping is only a function) and*
2. *the series*
$$\sum_{n=0}^{\infty} f_n$$
converges absolutely, i.e. $\sum_{n=0}^{\infty} |f_n| < +\infty$.

Conclusion

Proposition 118.
A (numeric) function

$$f : \mathbb{N}_0 \to \mathbb{R}, n \mapsto f_n.$$

is integrable if and only if the corresponding series is absolutely convergent i.e. if and only if $\sum_{n=0}^{\infty} |f_n| < +\infty$.
If this condition if satisfied, we have

$$\int_{\mathbb{N}_0} f_n \, d\mu(n) = \sum_{n=0}^{+\infty} f_n.$$

Proof. For absolutely convergent series, one may change the order of summation. Thus

$$\begin{aligned}
\int_{\mathbb{N}_0} f_n \, d\mu(n) &= \int_{\mathbb{N}_0} f_n^+ \, d\mu(n) - \int_{\mathbb{N}_0} f_n^- \, d\mu(n) \\
&= \sum_{n=0}^{\infty} \max\{f_n, 0\} - \sum_{n=0}^{\infty} \max\{-f_n, 0\} \\
&= \sum_{n=0}^{+\infty} f_n.
\end{aligned}$$

□

Example 119.

The function

$$f : \mathbb{N}_0 \to \mathbb{R}, n \mapsto f_n := \frac{1}{n}$$

is not integrable, but it is quasi-integrable with

$$\int_{\mathbb{N}_0} \frac{1}{n} \, d\mu(n) = \sum_{n=0}^{\infty} \frac{1}{n} = +\infty.$$

The 'alternating' function $g : \mathbb{N}_0 \to \mathbb{R}$ with $g_n := (-1)^n \cdot \frac{1}{n}$ is thus not integrable, even if the limit

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n} = \lim_{m\to\infty} \sum_{n=0}^{m} (-1)^n \cdot \frac{1}{n}$$

exists (conditional convergence of the series!).

2.6. Comparing the Lebesgue integral and the Riemannian one

The settings of the problem

Let us consider a continuous function defined on a closed interval:

$$f : [a, b] \to \mathbb{R}, x \mapsto f(x).$$

If we extend f by 0 outside of the closed interval $[a, b]$, we get a measurable function defined all over the real line; we denote this function by f again.

We can compute the integral of f over the closed interval $[a, b]$ in two ways:

1. as a Riemannian integral (R)-$\int f(x) \, dx$ and
2. as a Legesgue integral $\int_{[a,b]} f(x) \, d\lambda^1(x)$.

Our aim: comparing these two integrals

In order to compare these two integrals, we define two new functions:

1. For $x \in [a, b]$, we put

$$F(x) := \int_{[a,x]} f(\xi) \, d\lambda^1(\xi) = \int_{\mathbb{R}} \chi_{[a,x]}(\xi) \cdot f(\xi) \, d\lambda^1(\xi).$$

2. For $x \in [a,b]$ again, we put

$$G(x) := \text{(R)-}\int_a^x f(\xi)\,d\xi.$$

Remark that

$$F(a) = G(a) = 0.$$

The derivative of G

By the main theorem of the differential and integral calculus, we know that G has a derivative on $[a,b]$ (with one-sided derivatives on the border):

$$G'(x) = f(x).$$

The derivative of F

For $h > 0$ and small, we have

$$\begin{aligned} F(x+h) - F(x) &= \int_{\mathbb{R}} \chi_{[a,x+h]}(\xi) \cdot f(\xi)\,d\lambda^1(\xi) - \\ &\quad - \int_{\mathbb{R}} \chi_{[a,x]}(\xi) \cdot f(\xi)\,d\lambda^1(\xi) \\ &= \int_{\mathbb{R}} \chi_{[x,x+h]}(\xi) \cdot f(\xi)\,d\lambda^1(\xi). \end{aligned}$$

Note that we have used the fact that $\lambda^1(\{x\}) = 0$.

We remark that

$$h \cdot f(x) = f(x) \cdot \int_{\mathbb{R}} \chi_{[x,x+h]}(\xi)\,d\lambda^1(\xi) = \int_{\mathbb{R}} f(x) \cdot \chi_{[x,x+h]}(\xi)\,d\lambda^1(\xi)$$

Moreover, given a tolerance $\varepsilon > 0$, we can determine a threshold $\delta > 0$ in such a way that

$$|f(\xi) - f(x)| < \varepsilon, \qquad \forall \xi \in [x, x+\delta].$$

Putting these remarks together, we get, as long as $0 < h < \delta$,

$$\begin{aligned} F(x+h) - F(x) - h \cdot f(x) &= \int_{\mathbb{R}} (f(\xi) - f(x)) \cdot \chi_{[x,x+h]}(\xi)\,d\lambda^1(\xi) \\ |F(x+h) - F(x) - h \cdot f(x)| &\leq \int_{\mathbb{R}} \underbrace{|f(\xi) - f(x)|}_{<\varepsilon} \cdot \chi_{[x,x+h]}(\xi)\,d\lambda^1(\xi) \\ &< \varepsilon \cdot h \\ \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| &< \varepsilon \qquad \text{, if } 0 < h < \delta. \end{aligned}$$

A similar computation shows that

$$\left|\frac{F(x-h)-F(x)}{-h}-f(x)\right|<\varepsilon \qquad \text{, if } 0<h<\delta.$$

Thus we get

$$F'(x)=f(x).$$

Hence we may conclude that

$$F(x)\equiv G(x) \qquad (x\in[a,b]).$$

For continuous functions, the Riemannian and the Lebesgue integral coincide

Proposition 120.
Let $f:[a,b]\to\mathbb{R}$ be a continuous function.
Then

$$\int_a^b f(x)\,d\lambda^1(x) = \text{(R)-}\int_a^b f(x)\,dx.$$

Remark 121. *Thus, for continuous functions over a closed interval, all the integration techniques developed for the Riemannian integral remain valid for the Lebesgue integral.*

An important final remark

The above result is valid for integrals over a bounded interval.
As we will see later, for integrals like

$$\int_a^{+\infty} f(x)\,d\lambda^1(x)$$

the above result that the Riemannian and the Lebesgue integral coincide remains valid, if the continuous function f is absolutely integrable, i.e. if

$$\text{(R)-}\int_a^{+\infty} |f(x)|\,dx<+\infty.$$

If the convergence of the integral

$$\text{(R)-}\int_a^{+\infty} f(x)\,dx$$

is only conditional, the notions of Riemann and Legesgue may be different!

3

Convergence theorems

The aim of this chapter

The aim of this chapter is to give simple or weak conditions under which we have

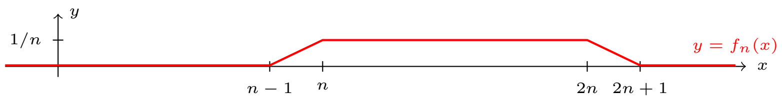
$$\int_X \lim_{n\to\infty} f_n(x)\, d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)\, d\mu(x).$$

Remark that 'uniform convergence' is not enough in order to exchange the order of integrals and limits, even if we assume that all functions f_n are continuous. The following example illustrates this.

$\int_X \lim_{n\to\infty} f_n(x)\, d\mu(x) \neq \lim_{n\to\infty} \int_X f_n(x)\, d\mu(x)$ is possible

Example 122.
Consider the sequence of functions $\{f_n\}_{n=1}^{+\infty}$ given by

$$f_n : \mathbb{R} \to \mathbb{R}, x \mapsto f_n(x) = \begin{cases} 0 & \text{, if } n \in]-\infty, n-1] \\ 1/n & \text{, if } n \le x \le 2n \\ 0 & \text{, if } [2n+1, +\infty[\\ \text{linear} & \text{, elsewhere.} \end{cases}$$



We have

$$\frac{1}{n}\chi_{[n,2n]}(x) \le f_n(x) \le \frac{1}{n}\chi_{[n-1,2n+1]}(x), \qquad \forall x \in \mathbb{R}.$$

Thus

$$1 \le \int_{\mathbb{R}} f_n(x)\, d\lambda^1(x) \le \frac{n+2}{n}$$

so that

$$\boxed{\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, \lambda^1(x) = 1.}$$

On the other hand,

$$\lim_{n\to\infty} f_n(x) = 0, \qquad \forall x \in \mathbb{R}$$

uniformly and

$$\boxed{\int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)\, d\lambda^1(x) = \int_{\mathbb{R}} 0\, d\lambda^1(x) = 0}$$

Thus we may conclude that

$$\boxed{\int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)\, d\lambda^1(x) \neq \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, d\lambda^1(x)}$$

for the present case.

3.1. Monotone convergence

Monotone convergence theorem by B. Levi in 1906

Theorem 123.

Hyp *Consider a sequence of numeric functions* $\{f_n(x)\}_{n=1}^{+\infty}$ *in* $\overline{\mathscr{L}}^+(X, \mathscr{A})$ *and suppose that*

- $f_n(x) \geq 0$ μ*-a.e. and that*
- *this sequence converges in a non-decreasing way:*

$$f_n(x) \nearrow \lim_{n\to\infty} f_n(x).$$

Concl *Limits and integration can be interchanged:*

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu(x) = \int_X \lim_{n\to\infty} f_n(x)\, d\mu(x)$$

Remark 124. *Remark that in the above theorem*

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu(x) = \int_X \lim_{n\to\infty} f_n(x)\, d\mu(x) = +\infty$$

is possible

Remark 125. *In short, we can say that*

- *positivity and*
- *monotonicity*

are sufficient conditions to exchange the order of an integral and a limit.

Proof. First of all remark that

$$f(x) := \lim_{n\to\infty} f_n(x) \in \overline{\mathscr{L}}(X, \mathscr{A}),$$

so that $\int_X f(x)\ d\mu(x) = \int_X \lim_{n\to\infty} f_n(x)\ d\mu(x) \in [0, +\infty]$ makes sense.

Step 1: We show that $\lim_{n\to\infty} \int_X f_n(x)\ d\mu(x) \le \int_X f(x)\ d\mu(x)$**.**

We have

$$\forall x \in X, \quad \forall n \in \mathbb{N}^*, \qquad f_n(x) \le f(x)$$

so that

$$\forall n \in \mathbb{N}^*, \qquad \int_X f_n(x)\ d\mu(x) \le \int_X f(x)\ d\mu(x).$$

Thus we get

$$\lim_{n\to\infty} \int_X f_n(x)\ d\mu(x) \le \int_X f(x)\ d\mu(x).$$

Step 2: We show that, $\forall u \in \mathscr{T}^+(X, \mathscr{A})$ **with** $u \le f$**, we have** $\int_X u(x)\ d\mu(x) \le \lim_{n\to\infty} \int_X f_n(x)\ d\mu(x)$

Fix some $\beta > 1$ and consider, for $n = 1, 2, 3, \ldots$, the sets

$$B_n := \{x \in X\ :\ \beta \cdot f_n(x) \ge u(x)\}.$$

Then

- $B_n \in \mathscr{A}$ since f_n and u are μ-measurable and since $B_n = \{x \in X\ :\ \beta \cdot f_n(x) - u(x) \ge 0\}$;
- $B_n \subset B_{n+1}$ ($n = 1, 2, 3, \ldots$);
- $B_n \nearrow X$ as $n \to \infty$ since $\lim_{n\to\infty} \beta \cdot f_n(x) = \beta \cdot f(x) \ge \beta \cdot u(x) > u(x)$ if $u(x) \neq 0$ and
- $\beta \cdot f_n(x) \ge u(x) \cdot \chi_{B_n}(x)$ (for all $x \in X$).

Thus we get, for all $\beta > 1$,

$$\int_X u(x) \cdot \chi_{B_n}(x)\ d\mu(x) \le \beta \cdot \int_X f_n(x)\ d\mu(x)$$

and

$$\lim_{n\to\infty} \int_X u(x) \cdot \chi_{B_n}(x)\ d\mu(x) \le \beta \cdot \lim_{n\to\infty} \int_X f_n(x)\ d\mu(x).$$

Making $\beta \to 0^+$, we get

$$\lim_{n\to\infty} \int_X u(x) \cdot \chi_{B_n}(x)\ d\mu(x) \le \lim_{n\to\infty} \int_X f_n(x)\ d\mu(x),$$

If we can show that $\lim_{n\to\infty} \int_X u(x) \cdot \chi_{B_n}(x)\ d\mu(x) = \int_X u(x)\ d\mu(x)$, we are done! This last equality can be shown as follows. We have

$$u(x) \cdot \chi_{B_n}(x) \in \mathscr{T}^+(X, \mathscr{A}) \quad \text{with} \quad u(x) \cdot \chi_{B_n}(x) \nearrow u(x) \quad \text{as } n \to \infty.$$

and hence we get

$$\lim_{n\to\infty}\int_X u(x)\cdot\chi_{B_n}(x)\,d\mu(x)=\int_X u(x)\,d\mu(x).$$

Thus we get the desired inequality

$$\int_X u(x)\,d\mu(x)\le\lim_{n\to\infty}\int_X f_n(x)\,d\mu(x).$$

Step 3: We show that $\int_X f(x)\,d\mu(x)\le\lim_{n\to\infty}\int_X f_n(x)\,d\mu(x)$
Let us consider a sequence $\{u_k\}_{k=1}^{+\infty}$ in $\mathscr{T}^+(X,\mathscr{A})$ with $u_k\nearrow f$ as $k\to\infty$. Then

- By Step 2 we have, for all $k=1,2,3,\ldots$,
$$\int_X u_k(x)\,d\mu(x)\le\lim_{n\to\infty}\int_X f_n(x)\,d\mu(x);$$
- Moreover
$$\lim_{k\to\infty}\int_X u_k(x)\,d\mu(x)=\int_X f(x)\,d\mu(x).$$

Thus we get, together with Step 1,

$$\int_X f(x)\,d\mu(x)\le\lim_{n\to\infty}\int_X f_n(x)\,d\mu(x)\le\int_X f(x)\,d\mu(x).$$

This gives the claim!

□

Integration of positive series

As an important corollary we have

Corollary 126.
Consider a sequence of non-negative functions $\{g_n(x)\}_{n=1}^{+\infty}$ *in* $\overline{\mathscr{Z}}^+(X,\mathscr{A})$*.*
Then

$$\int_X\sum_{n=1}^{\infty}g_n(x)\,d\mu(x)=\sum_{n=1}^{\infty}\int_X g_n(x)\,d\mu(x)$$

Proof. Put $f_n(x):=\sum_{k=1}^{n}g_k(x)$ and apply the monotone convergence theorem. □

Double sums of positive terms

Proposition 127.

Hyp *Consider* $\{a_{kn}\}$ *with* $k, n \in \{0, 1, 2, 3, \ldots\}$ *and suppose that*

$$a_{kn} \geq 0 \qquad \forall k, n.$$

Concl *The order of the sums can be interchanged:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{kn} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_{kn}.$$

Proof. Recalling that integrals over $(\mathbb{N}_0, \mathscr{P}(\mathbb{N}_0), \mu)$ with $\mu(A) = |A|$ are series, we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{kn} = \int_{\mathbb{N}_0} \sum_{k=0}^{\infty} a_{kn}\, d\mu(n) = \sum_{k=0}^{\infty} \int_{\mathbb{N}_0} a_{kn}\, d\mu(n) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_{kn}.$$

□

An integrability check for continuous functions

Proposition 128.
A continuous function

$$f : [a, +\infty[\to \mathbb{R}$$

is λ^1*-integrable if and only if*

$$\text{(R)-}\int_a^{+\infty} |f(x)|\, dx = \lim_{b \to +\infty} \text{(R)-}\int_a^b |f(x)|\, dx < +\infty.$$

Proof. For any sequence $\{b_n\}_{n=1}^{+\infty}$ with $\lim_{n\to\infty} b_n = +\infty$ we have

$$\begin{aligned}
\int_{[a,+\infty[} |f(x)|\, d\lambda^1(x) &= \lim_{n\to\infty} \int_{[a,+\infty[} \chi_{[a,b_n]} \cdot |f(x)|\, d\lambda^1(x) \\
&= \lim_{n\to\infty} \text{(R)-}\int_a^{b_n} |f(x)|\, dx \\
&= \text{(R)-}\int_a^{+\infty} |f(x)|\, dx.
\end{aligned}$$

□

Application of the integrability check for continuous functions

Example 129.

Let us show that the continuous function

$$f : [1, +\infty[\to \mathbb{R}, x \mapsto f(x) := \frac{\sin(\pi x)}{x^2}$$

is λ^1-integrable, i.e.

$$\int_{[1,+\infty[} |f(x)| \, d\lambda^1(x) = \int_{\mathbb{R}} \chi_{[1,+\infty[}(x) \cdot |f(x)| \, d\lambda^1(x) < +\infty.$$

In order to show this, we construct a sequence $\{g_n(x)\}_{n=1}^{+\infty}$ in $\overline{\mathscr{L}}^+(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with

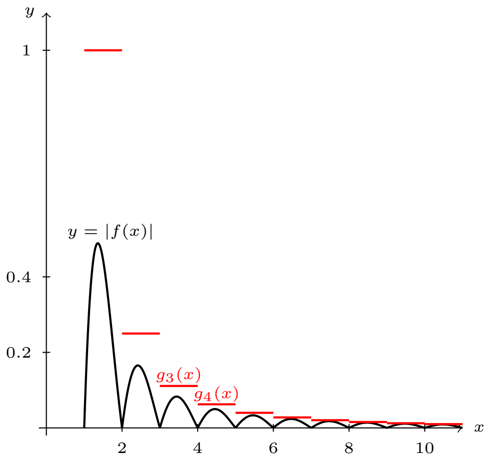
$$|f(x)| \leq \sum_{n=1}^{\infty} g_n(x), \qquad \forall x \in \mathbb{R}$$

and

$$\int_{\mathbb{R}} \sum_{n=1}^{\infty} g_n(x) \, d\lambda^1(x) = \sum_{n=1}^{\infty} \int_{\mathbb{R}} g_n(x) \, d\lambda^1(x) < +\infty.$$

Put

$$g_n(x) = \begin{cases} 1/n^2 & \text{, for } x \in [n, n+1[\\ 0 & \text{, elsewhere.} \end{cases}$$



Then

- For all $x \geq 1$,

$$|f(x)| \leq \sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \chi_{[n,n+1[}(x).$$

- Moreover

$$\int_{[1,+\infty[} \sum_{n=1}^{\infty} g_n(x)\, d\lambda^1(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{[1,+\infty[} \chi_{[n,n+1[}\, d\lambda^1 = \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty.$$

Thus f is λ^1-integrable over $[1,\infty[$.

3.2. Fatou's lemma

An extension of the monotone convergence theorem: Fatou's lemma

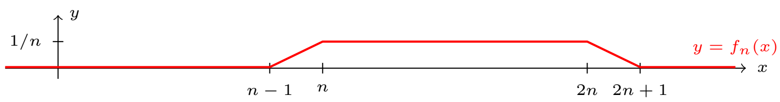
The example given above with

$$f_n : \mathbb{R} \to \mathbb{R}, x \mapsto f_n(x) = \begin{cases} 0 & \text{, if } n \in]-\infty, n-1] \\ 1/n & \text{, if } n \leq x \leq 2n \\ 0 & \text{, if } [2n+1, +\infty[\\ \text{linear} & \text{, elsewhere.} \end{cases}$$

shows that

$$\int_X \lim_{n\to\infty} f_n\, d\mu < \lim_{n\to\infty} \int_X f_n\, d\mu$$

is possible.



Fatou's lemma below shows that an inequality in the other direction can never occur if the functions f_n are non-negative.

Theorem 130.

Hyp Consider a sequence of non-negative functions $\{f_n(x)\}_{n=1}^{+\infty}$ in $\mathscr{L}^+(X, \mathscr{A})$.

Concl Then

$$\int_X \liminf_{n\to\infty} f_n\, d\mu \leq \liminf_{n\to\infty} \int_X f_n\, d\mu$$

The proof of Fatou's lemma

Proof. We put

$$f(x) := \liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} \inf_{k\geq n} f_k(x) \quad (x \in X)$$

and we consider, for $n = 1, 2, 3, \ldots$, the functions

$$g_n(x) := \inf_{k\geq n} f_k(x) \quad (x \in X).$$

Remark that

$$f \in \overline{\mathscr{L}}^+(X, \mathscr{A}) \qquad \text{and} \qquad g_n(x) \in \overline{\mathscr{L}}^+(X, \mathscr{A}) \quad (\text{for } n = 1, 2, 3, \ldots).$$

and that

$$g_n \nearrow f \quad \text{as } n \to \infty.$$

Thus, by the monotone convergence theorem,

$$\begin{aligned}
\int_X f(x)\, d\mu(x) &= \lim_{n\to\infty} \int_X g_n(x)\, d\mu(x) = \lim_{n\to\infty} \underbrace{\int_X \underbrace{\inf_{k\leq n} f_k(x)}_{\leq f_k(x) \text{ for } k\geq n} \quad d\mu(x)}_{\leq \int_X f_k(x)\, d\mu(x) \text{ for } k\geq n} \\
&\qquad\qquad \leq \inf_{k\geq n} \int_X f_k(x)\, d\mu(x) \\
&\leq \lim_{n\to\infty} \inf_{k\geq n} \int_X f_k(x)\, d\mu(x) \\
\int_X \liminf_{n\to\infty} f_n(x)\, d\mu(x) &\leq \liminf_{n\to\infty} \int_X f_n(x)\, d\mu(x).
\end{aligned}$$

□

3.3. Lebesgue's Dominated convergence theorem

Lebesgue's dominated convergence theorem

Theorem 131.

Hyp Let $(X, \mathscr{A}, \mu)$ be a measure space and suppose that $\{f_n(x)\}_{n=1}^{+\infty}$ and $f(x)$ are in $\overline{\mathscr{Z}}(X, \mathscr{A})$ and such that

- $\lim_{n\to\infty} f_n(x) = f(x)$ μ-a.e. on X;
- $\exists g(x) \in \mathscr{L}^1(X, \mathscr{A}, \mu)$ with

$$|f_n(x)| \leq g(x) \ \mu\text{-a.e. on } X, \quad \forall n \in \{1, 2, 3, \dots\}.$$

(g is called a majoration *or a* majorating function.*)*

Concl

1. f_n for $n \in \{1, 2, 3, \dots\}$ and f belong to $\mathscr{L}^1(X, \mathscr{A}, \mu)$.

2. Limit and integration can be interchanged:

$$\int_X \underbrace{\lim_{n\to\infty} f_n(x)}_{=f(x)} \ d\mu(x) = \lim_{n\to\infty} \int_X f_n(x) \ d\mu(x).$$

3. We have convergence in the L^1-norm, i.e.

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)| \ d\mu(x) = 0.$$

Proof. **Concerning the first point:**
Since $|f_n(x)| \leq g(x)$ μ-a.e., with $g \in \mathscr{L}^1(X, \mathscr{A}, \mu)$, we have

$$|f(x)| \leq g(x) \ \mu\text{-a.e.} \qquad \text{and} \qquad \int_X |f(x)| \, d\mu(x) \leq \int_X g(x) \ d\mu(x) < +\infty$$

so that $f \in \mathscr{L}^1(X, \mathscr{A}, \mu)$.
Concerning the second point:
This follows from the third point since

$$\left| \int_X f_n(x) \ d\mu(x) - \int_X f(x) \ d\mu(x) \right| \leq \int_X |f_n(x) - f(x)| \ d\mu(x) \to 0 \qquad (n \to \infty).$$

Concerning the last point:
Changing if necessary the values of the functions on null-sets, we may assume that (see the first point!)

- f and all f_n belong to $\overline{\mathscr{Z}}(X, \mathscr{A}, \mu)$ since

$$\{x \in X \ : \ f(x) \in \{\pm\infty\}\} \qquad \text{and} \qquad \{x \in X \ : \ f_n(x) \in \{\pm\infty\}\}$$

are all μ-null-sets.

- $\forall x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Consider the sequence $\{g_n\}_{n=1}^{+\infty}$ in $\overline{\mathscr{L}}(X, \mathscr{A}, \mu)$ given by

$$g_n(x) := |f(x)| + g(x) - |f(x) - f_n(x)|, \qquad (x \in X).$$

Since $|f(x) - f_n(x)| \leq |f(x)| + g(x)$ for all $x \in X$, we even have that $g_n(x) \in \overline{\mathscr{L}}^{+}(X, \mathscr{A}, \mu)$.
Moreover, by our hypotheses, we have

$$\lim_{n\to\infty} g_n(x) = |f(x)| + g(x) \qquad (x \in X).$$

Thus we may apply Fatou's lemma and we get

$$\begin{aligned}
\underbrace{\int_X [|f(x)| + g(x)]}_{\in\mathbb{R}} &= \int_X \underbrace{\lim_{n\to\infty} g_n(x)}_{=\liminf_{n\to\infty} g_n(x)} \quad d\mu(x) \\
&\leq \liminf_{n\to\infty} \int_X \underbrace{g_n(x)}_{=|f(x)|+g(x)-|f(x)-f_n(x)|} \quad d\mu(x) \\
&= \int_X [|f(x)| + g(x)] - \limsup_{n\to\infty} \int_X |f(x) - f_n(x)| \, d\mu(x)
\end{aligned}$$

i.e.

$$\lim_{n\to\infty} \int_X |f(x) - f_n(x)| \, d\mu(x) = 0.$$

□

3.4. Applications of the dominated convergence theorem

Topics where dominated convergence theorem may be applied

- the integration of series of functions;
- the continuous dependence on parameters in integrals;
- the differentiation with respect to a parameter in an integral;
- the Fourier transform.

The integration of series of functions

We would like to have

$$\int_X \sum_{k=1}^{\infty} g_k(x)\ d\mu(x) = \sum_{k=1}^{\infty} \int_X g_k(x)\ d\mu(x).$$

This situation can be covered by the dominated convergence theorem, since we would like to have

$$\int_X \lim_{n\to\infty} \sum_{k=1}^{n} g_k(x)\ d\mu(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \int_X g_k(x)\ d\mu(x).$$

Thus we put

$$f_n(x) := \sum_{k=1}^{n} g_k(x) \qquad \text{and} \qquad f(x) := \sum_{k=1}^{\infty} g_k(x)$$

We must formulate now hypotheses on $g_k(x)$ in such a way that we can apply the dominated convergence theorem. We need 3 facts

1. f_n and $f \in \overline{\mathscr{Z}}(X, \mathscr{A})$;
2. a majorating function $g \in \mathscr{L}^1(X, \mathscr{A}, \mu)$ with $f_n(x) \leq g(x)$ μ-a.e.
3. the limit $\lim_{n\to\infty} f_n(x) = f(x)$ must exist μ-a.e.

In order to have f_n and f in $\overline{\mathscr{Z}}(X, \mathscr{A})$ we impose the condition that

$$g_k(x) \in \overline{\mathscr{Z}}(X, \mathscr{A}) \qquad (k \in \{1, 2, 3, \ldots\}).$$

In order to get a majoration g, we proceed as follows:
Since

$$|f_n(x)| = \left|\sum_{k=1}^{n} g_k(x)\right| \leq \sum_{k=1}^{n} |g_k(x)|\,,$$

we can choose as a majoration $g(x) := \sum_{k=1}^{\infty} |g_k(x)|$. In order to have $g \in \mathscr{L}^1(X, \mathscr{A}, \mu)$, i.e.

$$\underbrace{\int_X \sum_{k=1}^{\infty} |g_k(x)|\ \mu(x)}_{\substack{=\sum_{k=1}^{\infty} \int_X |g_k(x)|\ \mu(x) \\ \text{by the monotone convergence theorem}}} < +\infty$$

we impose the condition that $g_k \in \mathscr{L}^1(X, \mathscr{A}, \mu)$ are such that

$$\sum_{k=1}^{\infty} \int_X |g_k(x)|\ \mu(x) < +\infty.$$

Remark that this condition implies that

$$\sum_{k=1}^{\infty} |g_k(x)| < +\infty \quad \mu\text{-a.e.}$$

so that the series

$$\sum_{k=1}^{\infty} g_k(x)$$

converges μ-a.e., i.e.

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \mu\text{-a.e.}$$

Hence we get

Proposition 132.
If the sequence $\{g_k(x)\}_{k=1}^{\infty}$ in $\mathscr{L}^1(X, \mathscr{A}, \mu)$ is such that

$$\sum_{k=1}^{\infty} \int_X |g_k(x)| \ \mu(x) < +\infty$$

then

$$\int_X \sum_{k=1}^{\infty} g_k(x)\, d\mu(x) = \sum_{k=1}^{\infty} \int_X g_k(x)\, d\mu(x).$$

Continuous dependence on parameters

In a measure space $(X, \mathscr{A}, \mu)$ we consider a function depending on a parameter:

$$f :]a, b[\times X \to \mathbb{R}, (\lambda, x) \mapsto f(\lambda, x).$$

We would like the function

$$F :]a, b[\to \mathbb{R}, \lambda \mapsto F(\lambda) := \int_X f(\lambda, x)\, d\mu(x)$$

to be continuous.

This situation can be covered by dominated convergence, since we would like to have

$$\lim_{\lambda\to\lambda_0} F(\lambda) = F(\lambda_0)$$

i.e. something like

$$\lim_{\lambda\to\lambda_0} \int_X f(\lambda, x)\, d\mu(x) = \int_X \lim_{\lambda\to\lambda_0} f(\lambda, x)\, d\mu(x) = \int_X f(\lambda_0, x)\, d\mu(x)$$

The last equality explains why we will impose the following condition:

$$f(\lambda, x) \text{ is continuous in } \lambda \ \mu\text{-a.e. on } X,$$

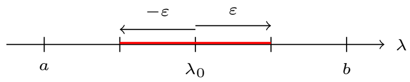
i.e.

$$\lim_{\lambda\to\lambda_0} f(\lambda, x) = f(\lambda_0, x) \qquad \mu\text{-a.e. on } X.$$

In order to apply the dominated convergence theorem, we need a majoration. This can be achieved as follows:
For a fixed $\lambda_0 \in]a, b[$, there exists a function $g \in \mathscr{L}^1(X, \mathscr{A}, \mu)$ with

$$|f(\lambda, x)| \leq g(x) \ \mu\text{-a.e. on } X, \quad \forall \lambda \in]\lambda_0 - \varepsilon, \lambda_0 + \varepsilon[\text{ for some } \varepsilon.$$

The above ε may thereby depend on the chosen λ_0.



Hence we get

Proposition 133.
Suppose that the function $f :]a, b[\times X \to \mathbb{R}$ is such that, for some fixed $\lambda_0 \in]a, b[$,

- $\lim_{\lambda \to \lambda_0} f(\lambda, x) = f(\lambda_0, x)$ *for μ-almost all $x \in X$;*
- *There exists a function $g \in \mathscr{L}^1(X, \mathscr{A}, \mu)$ with*

$$|f(\lambda, x)| \leq g(x) \ \mu\text{-a.e. on } X, \quad \forall \lambda \in]\lambda_0 - \varepsilon, \lambda_0 + \varepsilon[\text{ for some } \varepsilon.$$

Then the function

$$F :]a, b[\to \mathbb{R}, \lambda \mapsto F(\lambda) := \int_X f(\lambda, x) \, d\mu(x)$$

is continuous at λ_0.

Taking derivatives of integrals that depend on a parameter

In the context of the above example, we would like to take derivatives of F.
Hence we are interested in the following kind of computations (where $\{\lambda_n\}_{n=1}^{+\infty}$ is a sequence converging to some fixed $\tilde{\lambda}$ with $\lambda_n \neq \tilde{\lambda}$.)

$$\begin{aligned}
\lim_{n \to \infty} \frac{F(\lambda_n) - F(\tilde{\lambda})}{\lambda_n - \tilde{\lambda}} &= \lim_{n \to \infty} \int_X \frac{f(\lambda_n, x) - f(\tilde{\lambda}, x)}{\lambda_n - \tilde{\lambda}} \, d\mu(x) \\
&= \int_X \lim_{n \to \infty} \frac{f(\lambda_n, x) - f(\tilde{\lambda}, x)}{\lambda_n - \tilde{\lambda}} \, d\mu(x) \\
&= \int_X \frac{\partial}{\partial \lambda} f(\tilde{\lambda}, x) \, d\mu(x)
\end{aligned}$$

Hence, we will make the following assumptions: Suppose that the function $f :]a, b[\times X \to \mathbb{R}$ is such that

- $f(\lambda,\cdot) \in \mathscr{L}^1(X,\mathscr{A},\mu)$, $\forall \lambda \in]a,b[$, so that

$$F(\lambda) := \int_X f(\lambda,x)\, d\mu(x)$$

  is well-defined;
- For some fixed $\tilde{\lambda} \in]a,b[$, the derivative $\frac{\partial}{\partial\lambda} f(\tilde{\lambda},x)$ exists for all $x \in X$.
- $\exists g \in \mathscr{L}^1(X,\mathscr{A},\mu)$ with

$$\left|\frac{f(\lambda,x)-f(\tilde{\lambda},x)}{\lambda-\tilde{\lambda}}\right| \leq g(x)\ \mu\text{-a.e.} \qquad \forall \lambda \in]\tilde{\lambda}-\varepsilon, \tilde{\lambda}+\varepsilon[$$

  for some $\varepsilon > 0$.

Remark that this last condition is satisfied if there exists a $g \in \mathscr{L}^1(X,\mathscr{A},\mu)$ with

$$\left|\frac{\partial}{\partial\lambda} f(\lambda,x)\right| \leq g(x) \qquad \mu\text{-a.e. on } X,$$

and if the partial derivative $\frac{\partial}{\partial\lambda} f(\lambda,x)$ is continuous in λ at $\tilde{\lambda}$. This follows form the following computation

$$\left|\frac{f(\lambda,x)-f(\tilde{\lambda},x)}{\lambda-\tilde{\lambda}}\right| = \left|\frac{\partial}{\partial\lambda} f(\overline{\lambda},x)\right|$$

for some $\overline{\lambda}$ that lies between λ and $\tilde{\lambda}$.

Proposition 134.

<u>*Hyp*</u> *The above assumptions*

<u>*Concl*</u> *The derivative $F'(\tilde{\lambda})$ exist and*

$$F'(\tilde{\lambda}) = \frac{d}{d\lambda}\int_X f(\lambda,x)\, d\mu(x)\Big|_{\lambda=\tilde{\lambda}} = \int_X \frac{d}{d\lambda} f(\tilde{\lambda},x)\, d\mu(x)$$

Applications to the Fourier Transform

We will discuss these applications somewhat later for the following reason. The Fourier-transform in L^1 is defined by

$$\mathscr{F}_1[f(x)](\lambda) = \int_{\mathbb{R}} f(x)\cdot e^{-2\pi i\lambda x}\, d\lambda^1(x) =: \hat{f}(\lambda).$$

We integrate thus a complex-valued function $f(x)\cdot e^{-2\pi i\lambda x}$. Thus we first must give a meaning to the integral of complex-valued functions.

3.5. Complex valued functions

When dealing with complex valued funtions

$$f : X \to \mathbb{C}$$

over a measure space $(X, \mathscr{A}, \mu)$, we identify $\mathbb{C}$ with $\mathbb{R}^2$ and we equip the target space $\mathbb{C}$ with the σ-algebra $\mathscr{B}(\mathbb{C}) := \mathscr{B}(\mathbb{R}^2)$.

It turns out that a function is $\mathscr{A}$-measurable if and only if the real part $\Re f$ and the imaginary part $\Im f$ are $\mathscr{A}$-measurable.

For complex-valued, $\mathscr{A}$-measurable functions f we put

$$\int_X f(x)\ d\mu(x) := \int_X \Re f(x)\ d\mu(x) + i \cdot \int_X \Im f(x)\ d\mu(x)$$

It turns out that

- f is μ-integrable $\Longleftrightarrow$ $\Re f$ and $\Im f \in \mathscr{L}^1(X, \mathscr{A}, \mu) \Longleftrightarrow |f| \in \mathscr{L}^1(X, \mathscr{A}, \mu)$.
- The dominated convergence theorem remains valid for complex valued functions.

3.6. Application to the Fourier transform

An important application is given by the Fourier transform. Let

$$f : \mathbb{R} \to \mathbb{C}, x \mapsto f(x) := \Re f(x) + i \cdot \Im f(x)$$

be $\mathscr{L}^1$-measurable (this is the case if f is for example continuous) and suppose that

$$\Re f, \Im f \in \mathscr{L}^1(X, \mathscr{A}, \lambda^1),$$

i.e.

$$f \in \mathscr{L}^1_{\mathbb{C}}(\mathbb{R}, \mathscr{L}^1, \lambda^1) = \left\{ f : \mathbb{R} \to \mathbb{C}\ :\ \begin{array}{c} f \text{ is } \mathscr{L}^1\text{-measurable} \\ \text{and } \int_{\mathbb{R}} |f(x)|\ d\lambda^1(x) < +\infty \end{array} \right\}.$$

Then the function

$$g(\lambda, x) := f(x) \cdot e^{-2\pi i \lambda x} \in \mathscr{L}^1_{\mathbb{C}}(\mathbb{R}, \mathscr{L}^1, \lambda^1) \qquad \forall \lambda \in \mathbb{R}$$

since $|g(\lambda, x)| = |f(x)|$.

Thus the Fourier transformed

$$\mathscr{F}_1\left[f(x)\right](\lambda) := \int_{\mathbb{R}} f(x) \cdot e^{-2\pi i \lambda x}\ d\lambda^1(x) =: \hat{f}(\lambda)$$

is well-defined.

Continuity of the Fourier transform on $\mathscr{L}^1$

Proposition 135.
For any $f \in \mathscr{L}^1(\mathbb{R}, \mathscr{B}(\mathbb{R}, \lambda^1)$, the Fourier transformed

$$\mathscr{F}_1[f(x)](\lambda) = \hat{f}(\lambda) := \int_{\mathbb{R}} f(x) e^{-2\pi i \lambda x} \, d\lambda^1(x)$$

is well-defined and continuous in λ.

Proof. The fact that $\hat{f}$ is well-defined follows from

$$\int_{\mathbb{R}} \left| f(x) e^{-2\pi i \lambda x} \right| \, d\lambda^1(x) = \int_{\mathbb{R}} |f(x)| \, d\lambda^1(x) < +\infty.$$

The continuity can be established by the use of the majoration $g(x) := |f(x)|$. □

The Fourier transform is bounded

Proposition 136.
We have

$$\mathscr{F}_1 : \mathscr{L}^1_{\mathbb{C}}(\mathbb{R}, \mathscr{L}^1, \lambda^1) \to C(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \ : \ f \text{ continuous}\}$$

and

$$\|\hat{f}\|_\infty := \sup_{\lambda \in \mathbb{R}} |\hat{f}(\lambda)| \leq \int_{\mathbb{R}} |f(x)| \, d\lambda^1(x) =: \|f\|_{L^1}.$$

i.e.

$$\|\mathscr{F}_1[f(x)](\cdot)\|_\infty \leq \|f\|_{L^1} \qquad (f \in \mathscr{L}^1_{\mathbb{C}}(\mathbb{R}, \mathscr{L}^1, \lambda^1)).$$

Differentiability of a Fourier transformed $\hat{f}$

The Fourier transformed

$$\hat{f}(\lambda) = \int_{\mathbb{R}} |f(x)| \, d\lambda^1(x)$$

of f is an integral depending on a parameter λ. We can take a derivative with respect to the parameter λ

$$\begin{aligned} \hat{f}'(\lambda) &= \frac{d}{d\lambda} \int_{\mathbb{R}} f(x) \cdot e^{-2\pi i \lambda x} \, d\lambda^1(x) = \int_{\mathbb{R}} f(x) \cdot \frac{d}{d\lambda} e^{-2\pi i \lambda x} \, d\lambda^1(x) \\ &= -2\pi i \int_{\mathbb{R}} x \cdot f(x) \cdot e^{-2\pi i \lambda x} \, d\lambda^1(x) \end{aligned}$$

under the following assumptions:

- $f \in \mathscr{L}^1_{\mathbb{C}}(\mathbb{R}, \mathscr{L}^1, \lambda^1)$;
- We have

$$\begin{aligned}\left| f(x) \cdot \frac{e^{-2\pi i\lambda x} - e^{-2\pi i\tilde{\lambda}x}}{\lambda - \tilde{\lambda}} \right| &= |f(x)| \cdot \left| \frac{d}{d\lambda} e^{-2\pi i\lambda x} \Big|_{\lambda=\lambda'} \right| \\ &\quad \text{, where } \lambda' \text{ is between } \lambda \text{ and } \tilde{\lambda} \\ &= |f(x)| \cdot \left| -2\pi i x e^{-2\pi i\lambda' x} \right| \\ &= 2\pi x \cdot |f(x)| \in \mathscr{L}^1(\mathbb{R}, \mathscr{L}^1, \lambda^1).\end{aligned}$$

Proposition 137.

Hyp *The function $f \in \mathscr{L}^1(\mathbb{R}, \mathscr{L}^1, \lambda^1)$ is such that $x \cdot f(x)$ is λ^1-integrable, i.e. such that*

$$\int_{\mathbb{R}} |x| \cdot |f(x)| \, d\lambda^1(x) < +\infty.$$

Concl *the Fourier transformed $\hat{f}(\lambda)$ is differentiable and*

$$\hat{f}'(\lambda) = \mathscr{F}_1[-2\pi i x \cdot f(x)](\lambda),$$

i.e.

$$\frac{d}{d\lambda}\mathscr{F}_1[f(t)](\lambda) = \mathscr{F}_1[-2\pi i x \cdot f(x)](\lambda).$$

4

Fubini's theorem

4.1. Interchanging the order of integration

Double integrals

Let us consider the universe $\mathbb{R}^2$ equipped with the Lebesgue measure λ^2. This measure is defined on the complete σ-algebra $\mathscr{L}^2$ containing the σ-algebra $\mathscr{B}(\mathbb{R}^2)$ generated by $\mathscr{J}^2$.

Recall that a $\mathscr{L}^2$-measurable (numeric) function (for example a continuous function) f is λ^2-integrable if and only if

$$\int_{\mathbb{R}^2} |f(x,y)|\, d\lambda^2(x,y) < +\infty.$$

Thereby the integral of a positive (numeric) function is defined as the limit

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} u_n(x,y)\, d\lambda^2(x,y),$$

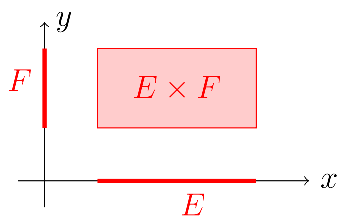
where the sequence $\{u_n\}_{n=1}^{+\infty}$ of step-function satisfying $u_n \nearrow f$.

For a given $\mathscr{L}^2$-measurable (numeric) function, we will consider in what follows integrals over λ^2-measurable sets $E \times F$. As typical examples of such sets let us mention the rectangle

$$]a,b]\times]c,d] \in \mathscr{J}^2, \quad \text{or} \quad [a,b]\times\mathbb{R} \quad \text{or} \quad \mathbb{R}^2 = \mathbb{R}\times\mathbb{R}.$$

For such sets, we have defined

$$\int_{E\times F} f(x,y)\, d\lambda^2(x,y) := \int_{\mathbb{R}^2} \chi_{E\times F}(x,y)\cdot f(x,y)\, d\lambda^2(x,y).$$



On may now fix some value $y \in F$ and consider the (numeric) function

$$f(\cdot,y) : E \to \overline{\mathbb{R}}, \quad x \mapsto f(x,y)$$

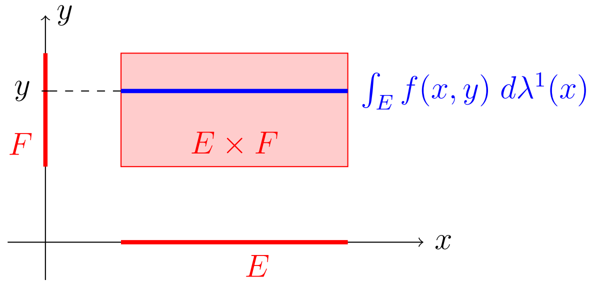
The are four interesting questions:

1. Is this (numeric) function $f(\cdot,y)$ λ^1-measurable (for λ^1-a.a. fixed y)?
2. If yes, is this function λ^1-integrable so that

$$\int_E f(x,y)\, d\lambda^1(x) \text{ makes sense (for } \lambda^1\text{-a.a. fixed } y)?$$

3. If yes, is the function $y \mapsto \int_E f(x,y)\, d\lambda^1(x) \quad \lambda^1$-measurable?
4. If yes, is this last function λ^1-integrable so that

$$\int_F \left[\int_E f(x,y)\, d\lambda^1(x)\right]\, d\lambda^1(y) \text{ makes sense?}$$



Let us introduce the notation

$$\int_F \int_E f(x,y)\, d\lambda^1(x)\, d\lambda^2(y) := \int_F \left[\int_E f(x,y)\, d\lambda^1(x)\right]\, d\lambda^1(y).$$

In a symmetric way we may consider, for a fixed $x \in E$, the (numeric) function

$$f(x,\cdot) : F \to \overline{\mathbb{R}}, \quad y \mapsto f(x,y)$$

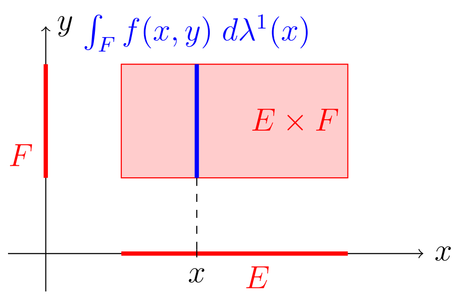
The are again four interesting questions:

1. Is this (numeric) function $f(x,\cdot)$ λ^1-measurable (for λ^1-a.a. fixed x)?
2. If yes, is this function λ^1-integrable so that

$$\int_F f(x,y)\, d\lambda^1(y) \text{ makes sense (for } \lambda^1\text{-a.a. fixed } x)?$$

3. If yes, is the function $x \mapsto \int_F f(x,y)\, d\lambda^1(y) \quad \lambda^1$-measurable?
4. If yes, is this last function λ^1-integrable so that

$$\int_E \left[\int_F f(x,y)\, d\lambda^1(y)\right]\, d\lambda^1(x) \text{ makes sense?}$$



Let us introduce the notation

$$\int_E \int_F f(x,y)\, d\lambda^1(y)\, d\lambda^2(x) := \int_E \left[\int_F f(x,y)\, d\lambda^1(y) \right] d\lambda^1(x).$$

There remains a final important question. Are the three integrals (if they exist)

$$\int_{E\times F} f(x,y)\, d\lambda^2(x,y), \quad \int_E \int_F f(x,y)\, d\lambda^1(y)\, d\lambda^1(x) \quad \text{and} \int_F \int_E f(x,y)\, d\lambda^1(x)\, d\lambda^1(y)$$

equal?

Fubini's theorem

Hyp

- The function $f : \mathbb{R}^2 \to \overline{\mathbb{R}}, \quad (x,y) \mapsto f(x,y)$ is $\mathscr{L}(\mathbb{R}^2)$-measurable
- The set $E \times F$ belongs to $\mathscr{L}(\mathbb{R}^2)$.

<u>Concl</u>

1. If the function f is non-negative on $E \times F$, then

$$\begin{aligned}\int_{E\times F} f(x,y)\, d\lambda^2(x,y) &= \int_E \int_F f(x,y)\, d\lambda^1(y)\, d\lambda^1(x)\\ &= \int_F \int_E f(x,y)\, d\lambda^1(x)\, d\lambda^1(y)\end{aligned}$$

   The three integrals may be equal to $+\infty$.

2. The function f is integrable, i.e.

$$\int_{E\times F} |f(x,y)|\, d\lambda^2(x,y) < +\infty$$

   if and only if

$$\int_E \int_F |f(x,y)|\, d\lambda^1(y)\, d\lambda^1(x) \quad \text{or} \quad \int_F \int_E |f(x,y)|\, d\lambda^1(x)\, d\lambda^1(y)$$

   is finite.

3. If f is integrable, then
   - the function $f(\cdot, y) : \quad x \mapsto f(x,y)$ is integrable for almost all y,
   - the function $f(x, \cdot) : \quad y \mapsto f(x,y)$ is integrable for almost all x and
   - the following three integrals exists in $\mathbb{R}$ and

$$\begin{aligned}\int_{E\times F} f(x,y)\, d\lambda^2(x,y) &= \int_E \int_F f(x,y)\, d\lambda^1(y)\, d\lambda^1(x)\\ &= \int_F \int_E f(x,y)\, d\lambda^1(x)\, d\lambda^1(y).\end{aligned}$$

4.2. The σ-algebra $\mathscr{A}_1 \otimes \mathscr{A}_2$

Our aim

Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ be two measure spaces and assume that both μ_1 and μ_2 are σ-finite.

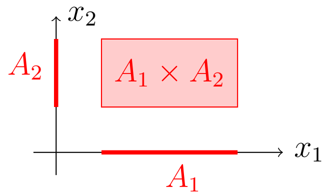
Our aim is to define a measure

$$\mu := \mu_1 \otimes \mu_2 \qquad \text{on } X := X_1 \times X_2$$

that agrees with the measures μ_1 and μ_2:

$$(\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2), \qquad \forall A_i \in \mathscr{A}_i \quad (i = 1, 2).$$

Example 138.
As a typical example we mention the case where $X_1 = X_2 = \mathbb{R}$, $\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{B}(\mathbb{R})$.



The notion of product σ-algebra $\mathscr{A}_1 \otimes \mathscr{A}_2$

Definition 139.
Given:

- measure spaces $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ with
- σ-finite measures μ_1 and μ_2

we define: the product σ-algebra $\mathscr{A}_1 \otimes A_2$ as:

$$\sigma\left(\{A_1 \times A_2 \ : \ A_i \in \mathscr{A}_1 \text{ for } i = 1, 2\}\right).$$

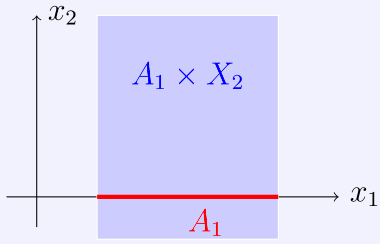
Thus $\mathscr{A}_1 \otimes \mathscr{A}_2$ is the smallest σ algebra containing all rectangles $A_1 \times A_2$ with $A_i \in \mathscr{A}_i$ for $i = 1, 2$.

Another generator for $\mathscr{A}_1 \otimes \mathscr{A}_2$

Proposition 140.

Hyp *Consider the family of "cylinders"*

$$\mathscr{C} := \{A_1 \times X_2 \ : \ A_1 \in \mathscr{A}_1\} \cup \{X_1 \times A_2 \ : \ A_2 \in \mathscr{A}_2\}$$



Concl $\mathscr{C}$ *is a generator for* $\mathscr{A}_1 \otimes \mathscr{A}_2$*, i.e.* $\sigma(\mathscr{C}) = \mathscr{A}_1 \otimes \mathscr{A}_2$.

Proof. 1. Since $\mathscr{C} \subset \mathscr{A}_1 \otimes \mathscr{A}_2$ (remark indeed that $X_i \in \mathscr{A}_i$), we have

$$\sigma(\mathscr{C}) \subset \mathscr{A}_1 \otimes \mathscr{A}_2.$$

2. On the other hand,

$$A_1 \times A_2 = (A_1 \times X_2) \cap (X_1 \times A_2)$$

gives

$$\{A_1 \times A_2 \ : \ A_1 \in \mathscr{A}_2 \text{ for } i = 1, 2\} \subset \sigma(\mathscr{C})$$

and thus

$$\mathscr{A}_1 \otimes \mathscr{A}_2 \subset \sigma(\mathscr{C}).$$

□

The main example

Example 141.
Let us consider the case where $X_1 = X_2 = \mathbb{R}$, $\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{B}(\mathbb{R})$.
Then

$$\boxed{\mathscr{A}_1 \otimes \mathscr{A}_2 = \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) = \mathscr{B}(\mathbb{R}^2).}$$

Indeed,

$$\mathscr{J}^1 \times \mathscr{J}^1 = \mathscr{J}^2,$$

so it is enough to show

$$\mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) = \underbrace{\sigma(\underbrace{\{B_1 \times B_2 \ : \ B_i \in \mathscr{J}^1 \text{ for } i = 1, 2\}}_{\mathscr{J}^1 \times \mathscr{J}^1 = \mathscr{J}^2})}_{\mathscr{B}(\mathbb{R}^2)}.$$

Step 1: We show that $\mathscr{B}(\mathbb{R}^2) \subset \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R})$.

This follows from

$$\mathscr{J}^1 \subset \mathscr{B}(\mathbb{R}) \implies \mathscr{J}^1 \times \mathscr{J}^1 = \mathscr{J}^2 \subset \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R})$$
$$\implies \underbrace{\sigma(\mathscr{J}^2)}_{=\mathscr{B}(\mathbb{R}^2)} \subset \underbrace{\sigma(\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}))}_{=\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R})}$$

Step 2: We show that $\mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) \subset \mathscr{B}(\mathbb{R}^2)$.
Indeed, remark that

$$B_1 \times X_2 = \cup_{n\in\mathbb{Z}} \underbrace{(B_1 \times]n, n+1])}_{\in \mathscr{J}^2} \in \mathscr{B}(\mathbb{R}^2), \qquad \forall B_1 \in \mathscr{J}^1$$

and

$$X_1 \times B_2 = \cup_{n\in\mathbb{Z}} \underbrace{(]n, n+1] \times B_2)}_{\in \mathscr{J}^2} \in \mathscr{B}(\mathbb{R}^2), \qquad \forall B_2 \in \mathscr{J}^1$$

so that $\mathscr{C} \subset \mathscr{B}(\mathbb{R}^2)$. But this implies

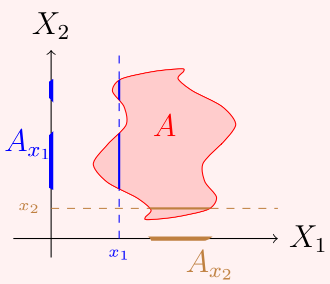
$$\mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) = \sigma(\mathscr{C}) \subset \mathscr{B}(\mathbb{R}^2).$$

The notion of sections of measurable sets

Definition 142.
For any $A \subset X_1 \times X_2$, we put

1. $$A_{x_1} := \{x_2 \in X_2 \ : \ (x_1, x_2) \in A\} \qquad (\subset X_2)$$

2. $$A_{x_2} := \{x_1 \in X_1 \ : \ (x_1, x_2) \in A\} \qquad (\subset X_1)$$



Sections of measurable sets are measurable

Theorem 143.

Hyp $A \in \mathscr{A}_1 \otimes \mathscr{A}_2$
Concl

1. $\forall x_1 \in X_1$, *we have* $A_{x_1} \in \mathscr{A}_2$;
2. $\forall x_2 \in X_2$, *we have* $A_{x_2} \in \mathscr{A}_1$.

Sections of measurable sets are measurable (proof)

Proof. We consider the sub-family $\mathscr{G}$ of $\mathscr{A}_1 \otimes \mathscr{A}_2$ consisting of the "good" measurable sets:

$$\mathscr{G} := \{A \in \mathscr{A}_1 \times \mathscr{A}_2 \ : \ \forall (x_1, x_2) \in X_1 \times X_2, \quad A_{x_1} \in \mathscr{A}_2 \text{ and } , A_{x_2} \in \mathscr{A}_1, \}$$

It is enough to show that $\mathscr{G} = \mathscr{A}_1 \otimes \mathscr{A}_2$.
We proceed in two steps:

- First we show that $\mathscr{G}$ is a σ-algebra;
- Then we show that $\mathscr{G}$ contains all rectangles $A_1 \times A_2$ with $A_i \in \mathscr{A}_i$ $(i = 1.2)$.

The conclusion follows then from

$$\mathscr{A}_1 \otimes \mathscr{A}_2 = \sigma\left(\{A_1 \times A_2 \ : \ A_i \in \mathscr{A}_1 \text{ for } i = 1,2\}\right) \subset \mathscr{G} \qquad (\subset \mathscr{A}_1 \otimes \mathscr{A}_2).$$

Step 1: $\mathscr{G}$ is a σ-algebra.
This follows from the following considerations:

- The whole space $X_1 \times X_2$ belongs to $\mathscr{G}$ since

$$\begin{aligned} &\forall x_1 \in X_1, \qquad && (X_1 \times X_2)_{x_1} = X_2 \in \mathscr{A}_2 \\ &\forall x_2 \in X_2, \qquad && (X_1 \times X_2)_{x_2} = X_1 \in \mathscr{A}_1. \end{aligned}$$

- $\mathscr{G}$ is $\complement$-stable since, $\forall A \in \mathscr{G}$, we have

$$\begin{aligned} &\forall x_1 \in X_1, \qquad && ((X_1 \times X_2) \setminus A)_{x_1} = X_2 \setminus \underbrace{(A_{x_1})}_{\in \mathscr{A}_2} \in \mathscr{A}_2 \\ &\forall x_2 \in X_2, \qquad && ((X_1 \times X_2) \setminus A)_{x_2} = X_1 \setminus \underbrace{(A_{x_2})}_{\in \mathscr{A}_1} \in \mathscr{A}_1. \end{aligned}$$

Thus it remains to show that $\mathscr{G}$ is $\cup_{n \in \mathbb{N}}$-stable.

- $\mathscr{G}$ is $\cup_{n\in\mathbb{N}}$-stable, since, for any collection $\{A_n\}_{n=1}^{+\infty}$ of set in $\mathscr{G}$, we have

$$\forall x_1 \in X_1, \qquad (\cup_{n=1}^{\infty} A_n)_{x_1} = \cup_{n=1}^{\infty} \underbrace{(A_n)_{x_1}}_{\in \mathscr{A}_2} \in \mathscr{A}_2$$
$$\forall x_2 \in X_2, \qquad (\cup_{n=1}^{\infty} A_n)_{x_2} = \cup_{n=1}^{\infty} \underbrace{(A_n)_{x_2}}_{\in \mathscr{A}_1} \in \mathscr{A}_1.$$

Step 2: $\mathscr{G}$ contains all rectangles $A_1 \times A_2$ with $A_i \in \mathscr{A}_i$ $(i = 1.2)$, since

$$\begin{aligned} \forall x_1 \in X_1, \qquad (A_1 \times A_2)_{x_1} &= \begin{cases} A_2 & \text{if } x_1 \in A_1 \\ \varnothing & \text{if } x_1 \notin A_1 \end{cases} \\ (A_1 \times A_2)_{x_1} &\in \mathscr{A}_2 \end{aligned}$$

and in a similar way

$$\forall x_2 \in X_2, \qquad (A_1 \times A_2)_{x_2} \in \mathscr{A}_2.$$

Thus we are done! □

4.3. The measure $\mu_1 \otimes \mu_2$

Let us draw a balance

Recall that for a "rectangle" $A_1 \times A_2$ (with $A_i \in \mathscr{A}_i$ for $i = 1, 2$) we would like to have

$$(\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2).$$

Moreover, we obtain now

$$\mu_1(A_{x_2}) = \begin{cases} \mu_1(A_1) & \text{if } x_2 \in \mathscr{A}_2 \\ 0 & \text{if } x_2 \notin \mathscr{A}_2 \end{cases}$$

i.e.

$$\mu_1(A_{x_2}) = \mu_1(A_1) \cdot \chi_{A_2}(x_2).$$

so that

$$\int_{X_2} \mu_1(A_{x_2})\, d\mu_2(x_2) = \mu_1(A_1) \cdot \mu_2(A_2) = (\mu_1 \otimes \mu_2)(A_1 \times A_2)$$

Proceeding in a symmetric way with $\mu_2(A_{x_1})$, we get

$$\begin{aligned} (\mu_1 \otimes \mu_2)(A_1 \times A_2) &= \int_{X_2} \mu_1(A_{x_2})\, d\mu_2(x_2) \\ &= \int_{X_1} \mu_2(A_{x_1})\, d\mu_1(x_1). \end{aligned}$$

This last result could be used to define the product measure $\mu_1 \otimes \mu_2$. In order to do this, we need the following result:

Proposition 144.
For all $A \in \mathscr{A}_1 \otimes \mathscr{A}_2$ we have

$$x_1 \mapsto \mu_2(A_{x_1}) \quad \text{is } \mathscr{A}_1\text{-measurable} \quad \text{and}$$
$$x_2 \mapsto \mu_1(A_{x_2}) \quad \text{is } \mathscr{A}_2\text{-measurable.}$$

Moreover

$$\int_{X_2} \mu_1(A_{x_2})\, d\mu_2(x_2) = \int_{X_1} \mu_1(A_{x_1})\, d\mu_1(x_1) \quad (\in [0, +\infty]).$$

Definition of $\mu_1 \otimes \mu_2$

Definition 145.
Given:

- measure spaces $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ with
- σ-finite measures μ_1 and μ_2

we define: product measure $\mu_1 \otimes \mu_2$ as:

$$(\mu_1 \otimes \mu_2)(A) := \int_{X_2} \mu_1(A_{x_2})\, d\mu_2(x_2) \tag{4.1}$$
$$= \int_{X_1} \mu_1(A_{x_1})\, d\mu_1(x_1) \quad (\in [0, +\infty]) \tag{4.2}$$

for all $A \in \mathscr{A}_1 \otimes \mathscr{A}_2$.

$\mu_1 \otimes \mu_2$ is a measure

Proposition 146.

Hyp

- *measure spaces $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ with*
- *σ-finite measures μ_1 and μ_2*

$\mu_1 \otimes \mu_2$ given by the above relation (4.1) or (4.2)

<u>Concl</u> *$\mu_1 \otimes \mu_2$ is a measure on $\mathscr{A}_1 \otimes \mathscr{A}_2$ with*

$$(\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2) \qquad \text{if } A_i \in \mathscr{A}_i \text{ for } i = 1, 2.$$

A typical example

Example 147.

We consider the case where $X_1 = X_2 = \mathbb{R}$ and $\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{B}(\mathbb{R})$. So

$$\mathscr{A}_1 \otimes \mathscr{A}_2 = \mathscr{B}(\mathbb{R}^2).$$

On X_1 and X_2, we consider the measures $\mu_1 = \mu_2 = \lambda^1|_{\mathscr{B}(\mathbb{R})} =: \beta^1$.

We have now two measures on $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$:

- $\beta^2 := \lambda^2|_{\mathscr{B}(\mathbb{R}^2)}$ and
- $\beta^1 \otimes \beta^1$.

Remark that both measures coincide on $\mathscr{J}^1 \times \mathscr{J}^1 = \mathscr{J}^2$ and that $\mathscr{J}^2$ is a generator for $\mathscr{B}(\mathbb{R}^2)$.

Recall that the extension by Carathéodry is unique.

Thus we get

$$\boxed{\beta^2 = \beta^1 \otimes \beta^1.}$$

Moreover,

$$\forall f \in \mathscr{L}^1(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \lambda^2), \qquad \int_{\mathbb{R}^2} f(x_1, x_2)\, d\beta^2(x_1, x_2) = \int_{\mathbb{R}^2} f(x_1, x_2)\, d(\beta^1 \otimes \beta^1)(x_1, x_2)$$

4.4. Integration with multiple integrals

Multiple integrals for characteristic functions

Let us first consider a characteristic function

$$f(x_1, x_2) := \chi_A(x_1, x_2) = \chi_{A_{x_1}(x_2)} = \chi_{A_{x_2}(x_1)}, \qquad \text{with } A \in \mathscr{A}_1 \otimes \mathscr{A}_2.$$

We have

$$\int_{X_1\times X_2} \underbrace{\chi_A(x_1,x_2)}_{=f(x_1,x_2)}\, d(\mu_1\otimes\mu_2)(x_1,x_2) = (\mu_1\otimes\mu_2)(A)$$

$$= \int_{X_1} \mu_2(A_{x_1})\, d\mu_1(x_1) = \int_{X_2} \mu_1(A_{x_2})\, d\mu_2(x_1)$$

$$= \begin{cases} \displaystyle\int_{X_1}\left[\int_{X_2} \underbrace{\chi_{A_{x_1}}(x_2)}_{=f(x_1,x_2)}\, d\mu_2(x_2)\right] d\mu_1(x_1) \\ \displaystyle\int_{X_2}\left[\int_{X_1} \underbrace{\chi_{A_{x_2}}(x_1)}_{=f(x_1,x_2)}\, d\mu_1(x_1)\right] d\mu_2(x_2) \end{cases}$$

Multiple integrals for step functions

By additivity, the above result remains true for step-functions:

Proposition 148.

Hyp $f \in \mathscr{T}(X_1\times X_2, \mathscr{A}_1\otimes\mathscr{A}_2)$, *where (for* $i=1,2$*)*

$(X_i,\mathscr{A}_i)$ *are measureable spaces that we equip with* σ*-finite measures* μ_i.

Concl *We have*

$$\int_{X_1\times X_2} f(x_1,x_2)\, d(\mu_1\otimes\mu_2)(x_1,x_2) = \begin{cases} \displaystyle\int_{X_1}\left[\int_{X_2} f(x_1,x_2)\, d\mu_2(x_2)\right] d\mu_1(x_1) \\ \displaystyle\int_{X_2}\left[\int_{X_1} f(x_1,x_2)\, d\mu_1(x_1)\right] d\mu_2(x_2) \end{cases}$$

Multiple integrals for positive, numeric functions (Tonelli)

Proposition 149.

Hyp $f \in \overline{\mathscr{L}}^+(X_1 \times X_2, \mathscr{A}_1 \otimes \mathscr{A}_2)$, *where (for* $i = 1, 2$*)*

$(X_i, \mathscr{A}_i)$ *are measureable spaces that we equip with* σ*-finite measures* μ_i*.*

Concl *We have*

$$\int_{X_1 \times X_2} f(x_1, x_2)\, d(\mu_1 \otimes \mu_2)(x_1, x_2) = \begin{cases} \displaystyle\int_{X_1} \left[\int_{X_2} f(x_1, x_2)\, d\mu_2(x_2) \right] d\mu_1(x_1) \\ \displaystyle\int_{X_2} \left[\int_{X_1} f(x_1, x_2)\, d\mu_1(x_1) \right] d\mu_2(x_2) \end{cases}$$

Corollary 150.

Hyp $f \in \overline{\mathscr{L}}(X_1 \times X_2, \mathscr{A}_1 \otimes \mathscr{A}_2)$, *where (for* $i = 1, 2$*)*

$(X_i, \mathscr{A}_i)$ *are measureable spaces that we equip with* σ*-finite measures* μ_i*.*

Concl f *is* $\mu_1 \otimes \mu_2$*-integrable, i.e.* $\int_{X_1 \times X_2} |f(x_1, x_2)|\, d(\mu_1 \otimes \mu_2)(x_1, x_2) < +\infty$ *if and only if on of the following integrals is finite*

$$\int_{X_1} \left[\int_{X_2} |f(x_1, x_2)|\, d\mu_2(x_2) \right] d\mu_1(x_1)$$

$$\text{or} \int_{X_1} \left[\int_{X_2} |f(x_1, x_2)|\, d\mu_2(x_2) \right] d\mu_1(x_1).$$

Fubini's theorem

Theorem 151.

<u>Hyp</u> $f \in \overline{\mathscr{L}}(X_1 \times X_2, \mathscr{A}_1 \otimes \mathscr{A}_2)$, *where (for* $i = 1, 2$*)*

$(X_i, \mathscr{A}_i)$ *are measureable spaces that we equip with* σ*-finite measures* μ_i.

<u>Concl</u> *If* f *is* $\mu_1 \otimes \mu_2$*-integrable, we have*

$$\int_{X_1 \times X_2} f(x_1, x_2)\, d(\mu_1 \otimes \mu_2)(x_1, x_2) = \\ = \begin{cases} \displaystyle\int_{X_1} \left[\int_{X_2} f(x_1, x_2)\, d\mu_2(x_2)\right] d\mu_1(x_1) \\ \\ \displaystyle\int_{X_2} \left[\int_{X_1} f(x_1, x_2)\, d\mu_1(x_1)\right] d\mu_2(x_2) \end{cases}$$

Moreover

- $\displaystyle\int_{X_2} f(x_1, x_2)\, d\mu_2(x_2)$ *if finite for* μ_1*-almost all* $x_1 \in X_1$ *and*
- $\displaystyle\int_{X_1} f(x_1, x_2)\, d\mu_1(x_1)$ *if finite for* μ_2*-almost all* $x_2 \in X_2$.

Remark 152. *If one applies the above theorem to the case*

$$X_1 = X_2 = \mathbb{R}, \qquad \text{with } \mathscr{A}_1 = \mathscr{A}_2 = \mathscr{B}(\mathbb{R}) \text{ and } \mu_1 = \mu_2 = \beta^1 := \lambda^1|_{\mathscr{B}(\mathbb{R})},$$

on has $\mathscr{A}_1 \otimes \mathscr{A}_2 = \mathscr{B}(\mathbb{R}^2)$ *and* $\beta^1 \otimes \beta^1 = \beta := \lambda^2|_{\mathscr{B}(\mathbb{R}^2)}$.

Remark however that

$$\mathscr{L}(\mathbb{R}) \otimes \mathscr{L}(\mathbb{R}) \subsetneq \mathscr{L}(\mathbb{R}^2)$$

and that

$$\lambda^1 \otimes \lambda^1 \text{ is not complete, the completion of } \lambda^1 \otimes \lambda^1 \text{ being } \lambda^2.$$

Fubini's theorem however remains valid:

Fubini's theorem (second version)

Theorem 153.

<u>Hyp</u> $f \in \mathscr{L}^1(\mathbb{R}^2, \mathscr{L}(\mathbb{R}^2), \lambda^2)$.

<u>*Concl*</u> *We have*

$$\int_{\mathbb{R}^2} f(x_1,x_2)\, d\lambda^2(x_1,x_2) = \begin{cases} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x_1,x_2)\, d\lambda^1(x_2) \right] d\lambda^1(x_1) \\ \\ \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x_1,x_2)\, d\lambda^1(x_1) \right] d\lambda^1(x_2) \end{cases}$$

Moreover

- $\int_{\mathbb{R}} f(x_1,x_2)\, d\lambda^1(x_2)$ *if finite for* λ^1*-almost all* $x_1 \in \mathbb{R}$ *and*
- $\int_{\mathbb{R}} f(x_1,x_2)\, d\lambda^1(x_1)$ *if finite for* λ^2*-almost all* $x_2 \in \mathbb{R}$.

Part II

Spaces with norms

5

Normed spaces

5.1. Linear spaces (vector spaces)

5.1.1. Definition and examples

A notations, we will use in what follows

We set, in what follows,

$$\mathbb{K} := \mathbb{R} \qquad \text{or} \qquad \mathbb{K} = \mathbb{C}.$$

The notion of linear space

Let us consider a non-empty set X that is equipped with

- an *addition:* $X \times X \to X$, $(u, v) \mapsto u + v$ and with
- a *multiplication by a scalar:* $\mathbb{K} \times X \to X$, $(\lambda, u) \mapsto \lambda \cdot u = \lambda u$.

Definition 154.
$(X, +, \cdot)$ a linear space (or a vector space):

We have

- $u + v = v + u$ for all u and $v \in X$;
- $(u + v) + w = u + (v + w)$ for all u, v and $w \in X$;
- $\exists! 0 \in X$ with $u + 0 = u$, $\forall u$;
- $\forall u \in X$, $\exists!(-u)$ with $u + (-u) = 0$.

Moreover, $\forall u$, $v \in X$ and $\forall \alpha$, $\beta \in \mathbb{K}$, we have

- $(\alpha + \beta)u = \alpha u + \beta u$;
- $\alpha(u + v) = \alpha u + \alpha v$;
- $\alpha(\beta u) = (\alpha\beta)u$;
- $1 \cdot u = u$.

A first example of a linear space

Example 155.
For $n = 1, 2, 3, \ldots$, we put

$$X := \mathbb{K}^N := \{x = (\xi_1, \xi_2, \ldots, \xi_N) \ : \ \xi_k \in \mathbb{K} \text{ for } k = 1, 2, \ldots, N\}.$$

and we equip $\mathbb{K}^N$ with

- the addition

$$(\xi_1, , \ldots, \xi_N) + (\eta_1, , \ldots, \eta_N) := (\xi_1 + \eta_1, \ldots, \xi_N + \eta_N);$$

- and the multiplication by a scalar

$$\alpha(\xi_1, \ldots, \xi_N) = (\alpha \cdot \xi_1, \ldots, \alpha \cdot \xi_N).$$

Then

Proposition 156.
$(\mathbb{K}^N, +, \cdot)$ *is a* $\mathbb{K}$*-linear space with*

- $0 = (0, \ldots, 0)$ *and*
- $-(\xi_1, \ldots, \xi_N) = (-\xi_1, \ldots, -\xi_N)$.

A second example of a linear space

Example 157.
Consider, for $-\infty < a < b < +\infty$ kept fixed, the set

$$C[a, b] := \{f : [a, b] \to \mathbb{K} \ : \ f \text{ is continuous}\}$$

equipped with

- the (pointwise) addition

$$(x + y)(t) := x(t) + y(t) \qquad (t \in [a, b])$$

- and the (pointwise) multiplication by a scalar

$$(\alpha \cdot x)(t) := \alpha \cdot x(t) \qquad (t \in [a, b])$$

Proposition 158.
$(C[a,b], +, \cdot)$ *is a* $\mathbb{K}$- *linear space with*

- $0(t) \equiv 0$ *and*
- $(-x)(t) \equiv -x(t)$.

The notion of linear independence

Definition 159.
Given: A $\mathbb{K}$-linear space X, and vectors

$$u_1, u_2, \dots u_n \in X.$$

we say: $u_1, u_2, \dots, u_n$ are linear independent iff:

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0 \Longrightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

Remark 160. *For* $N = 1, 2, 3, \dots$ *we write*

$$\dim X = N \qquad \text{(dimension of } X \text{ is } N\text{)}$$

if and only if the maximal number of linear independent elements in X *is* N.
We say that

$$\dim X = \infty$$

if and only if there exist N *linear independent elements in* X *for each* $N = 1, 2, 3, \dots$.

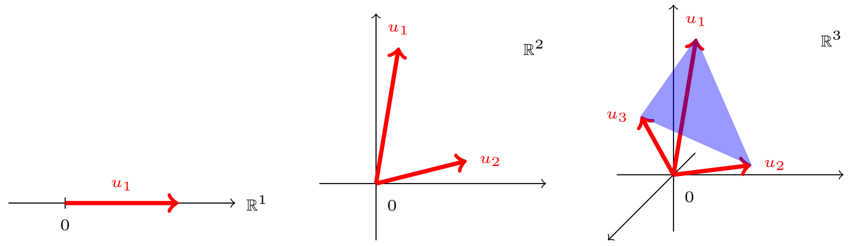
Remark 161. *We put*

$$\dim\{0\} = 0.$$

A 'typical' finite-dimensional linear space

Example 162.
We know that

$$\boxed{\dim \mathbb{K}^N = N \qquad (N = 1, 2, 3, \dots).}$$



The dimension of the space $C[a, b]$

Lemma 163.
We consider, in the linear space $C[a, b]$ (with $-\infty < a < b < +\infty$), the elements

$$u_k(x) := x^k \qquad \text{, for } k = 0, 1, 2, 3, \ldots.$$

Then, any set of elements of the form

$$u_0, u_1, u_2, \ldots, u_N \qquad \text{, where } N = 0, 1, 2, 3, \ldots \text{ is kept fixed}$$

is linear independent.

Proof. Indeed, the relation

$$\alpha_n u_1(x) + \cdots \alpha_N u_N(x) = 0 \qquad \forall x \in [a, b]$$

means that the polynomial

$$p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_N x^N$$

has an infinite number of zeros. But this is impossible unless $p(x)$ is the 0-polynomial, i.e. unless

$$\alpha_0 = \alpha_1 = \cdots = \alpha_N = 0.$$

□

Proposition 164.
The space $C[a, b]$ (where $-\infty < a < b < +\infty$) is infinite-dimensional:

$$\boxed{\dim C[a, b] = \infty.}$$

Corollary 165.
Any linear space X with $C[a,b] \subset X$ is infinite-dimensional, i.e.

$$\boxed{C[a,b] \subset X \Longrightarrow \dim X = \infty.}$$

Corollary 166.

$$\boxed{\dim \overline{\mathscr{Z}}(\mathbb{R}, \mathscr{L}) = \infty.}$$

Remark 167. *Thus, linear* function spaces *'usually' are infinite dimensional.*

5.1.2. The linear space $\mathscr{L}^p(X, \mathscr{A}, \mu)$

An additional 'convention'

For $p > 0$, we put

$$\boxed{(+\infty)^p = +\infty \qquad \text{and} \qquad (+\infty)^{-p} = 0.}$$

as a new convention.

Then, if $(X, \mathscr{A})$ is a measurable space, we have

$$\boxed{f \in \overline{\mathscr{Z}}(X, \mathscr{A}) \Longrightarrow |f|^p \in \overline{\mathscr{Z}}(x, \mathscr{A}) \qquad (p > 0).}$$

The notion of $\mathscr{L}^p$-spaces

Definition 168.
Given: A measure space $(X, \mathscr{A}, \mu)$ and a constant $p \geq 1$
we define: The Lebesgue spaces $\mathscr{L}^p(X, \mathscr{A}, \mu)$ and $\mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$ as:

$$\mathscr{L}^p(X, \mathscr{A}, \mu) := \{f : X \to \overline{\mathbb{R}} \quad ; \quad f \in \overline{\mathscr{Z}}(X, \mathscr{A}) \\ |f|^p \text{ is } \mu\text{-integrable}\}$$

and

$$\mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu) := \{f : X \to \mathbb{C} \quad : \quad \Im f, \Re f \in \mathscr{Z}(X, \mathscr{A}) \\ \text{and } |f|^p \text{ is } \mu\text{-integrable}\}$$

Remark 169. *In what follows, we will only deal with $\mathscr{L}^p(X, \mathscr{A}, \mu)$, but the derived results can be overtaken to $\mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$ if one replaces the absolute value $|\cdot|$ by the modulus $|\cdot|$.*

Remark 170. *For measurable functions we have*

$$\boxed{f \text{ is } \mu\text{-integrable} \iff f \in \mathscr{L}^1(X, \mathscr{A}, \mu).}$$

$|f|^p$ is μ-integrable means that

$$\int_X |f(x)|^p \, d\mu(x) < +\infty.$$

Thus

$$\boxed{\mathscr{L}^p(X, \mathscr{A}, \mu) := \left\{ f : X \to \overline{\mathbb{R}} \ : \ f \in \overline{\mathscr{Z}}(X, \mathscr{A}) \\ \int_X |f(x)|^p \, d\mu(x) < +\infty \right\}}$$

and

$$\boxed{\mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu) := \left\{ f : X \to \mathbb{C} \ : \ \Im f, \Re f \in \mathscr{Z}(X, \mathscr{A}) \\ \int_X |f(x)|^p \, d\mu(x) < +\infty \right\}.}$$

The notion of semi-norm $N_p(\cdot)$

We introduce the notation

$$N_p(f) := \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p} \qquad (p \geq 1).$$

Remark that is is possible to get

$$N_p(f) = +\infty,$$

but we have

$$\boxed{\mathscr{L}^p(X,\mathscr{A},\mu) := \left\{f \in \overline{\mathscr{Z}}(X,\mathscr{A})\ :\ N_p(f) < +\infty\right\}}$$

and

$$\boxed{\mathscr{L}^p_{\mathbb{C}}(X,\mathscr{A},\mu) := \left\{f : X \to \mathbb{C}\ :\ \Im f, \Re f \in \mathscr{Z}(X,\mathscr{A})\ N_p(f) < +\infty\right\}.}$$

Addition an multiplication by scalars in $\mathscr{L}^p$-spaces

We introduce now an addition and a multiplication by scalars on $\mathscr{L}^p(X,\mathscr{A},\mu)$. We invite the reader to treat the case $\mathscr{L}^p_{\mathbb{C}}(X,\mathscr{A},\mu)$ in parallel by himself,

We put

- $+ : \mathscr{L}^p(X,\mathscr{A},\mu) \times \mathscr{L}^p(X,\mathscr{A},\mu) \to \mathscr{L}^p(X,\mathscr{A},\mu)$ is given by pointwise addition:
$$(f+g)(x) := f(x) + g(x) \qquad (x \in X).$$
- $\cdot : \mathbb{R} \times \mathscr{L}^p(X,\mathscr{A},\mu) \to \mathscr{L}^p(X,\mathscr{A},\mu)$ is given by pointwise multiplication:
$$(\alpha \cdot f)(x) := \alpha \cdot f(x) \qquad (x \in X).$$

These definitions could be somewhat problematic, since for example it is not clear that

$$f, g \in \mathscr{L}^p(X,\mathscr{A},\mu) \Longrightarrow f + g \in \mathscr{L}^p(X,\mathscr{A},\mu).$$

or that

$$\alpha \in \mathbb{R}, f \in \mathscr{L}^p(X\mathscr{A},\mu) \Longrightarrow \alpha \cdot f \in \mathscr{L}^p(X\mathscr{A},\mu).$$

I fact we know that $f+g$ and $\alpha \cdot f$ are measurable functions, but we must check whether or not these functions belong to $\mathscr{L}^p(X,\mathscr{A},\mu)$.

This is easy for $\alpha \cdot f$:

$$\begin{aligned}
N_p(\alpha \cdot f) &= \left(\int_X |\alpha \cdot f(x)|^p\right)^{1/p} \\
&= \left(|\alpha|^p \int_X |f(x)|^p\right)^{1/p} \\
&= |\alpha| \left(\int_X |f(x)|^p\right)^{1/p} \\
&= |\alpha| \cdot N_p(f),
\end{aligned}$$

so

$$N_p(f) < +\infty \Longrightarrow N_p(\alpha \cdot f) < +\infty.$$

The following proposition shows that addition in $\mathscr{L}^p$-spaces is well defined, too.

Proposition 171.
Suppose that $1 \leq p < +\infty$. *Then*

$$f, g \in \mathscr{L}^p(X, \mathscr{A}, \mu) \Longrightarrow f + g \in \mathscr{L}^p(X, \mathscr{A}, \mu).$$

Proof. We have

$$\begin{aligned} |f(x) + g(x)|^p &\leq (|f(x)| + |g(x)|)^p \\ &\leq (2 \cdot \max\{|f(x)|, |g(x)|\})^p \\ &= 2^p \cdot \max\{|f(x)|^p, |g(x)|^p\} \\ &= 2^p \left(|f(x)|^p + |g(x)|^p\right) \end{aligned}$$

so that

$$\begin{aligned} \int_X |f(x) + g(x)|^p \, d\mu(x) &\leq 2^p \int_X \left(|f(x)|^p + |g(x)|^p\right) \, d\mu(x) \\ &= 2^p \cdot \int_X |f(x)|^p \, d\mu(x) + 2^p \cdot \int_X |g(x)|^p \, d\mu(x) < +\infty. \end{aligned}$$

□

$\mathscr{L}^p$-spaces are linear spaces for $p \in [1, +\infty[$
Thus we get

Proposition 172.
Let $p \in [1, +\infty[$ *be fixed. Then*

$$\mathscr{L}^p(X, \mathscr{A}, \mu) := \left\{ f : X \to \overline{\mathbb{R}} \ : \ f \in \overline{\mathscr{Z}}(X, \mathscr{A}), \quad \int_X |f(x)|^p \, d\mu(x) < +\infty \right\}$$

and

$$\mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu) := \left\{ f : X \to \mathbb{C} \ : \ \Re f, \Im f \in \mathscr{Z}(X, \mathscr{A}), \quad \int_X |f(x)|^p \, d\mu(x) < +\infty \right\}$$

when equipped with the above defined addition $f + g$ *and scalar multiplication* $\alpha \cdot f$ *are both linear spaces over* $\mathbb{R}$ *resp.* $\mathbb{C}$.

5.1.3. The linear space $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$

The notion of semi-norm $N_\infty(\cdot)$
Let $(X, \mathscr{A}, \mu)$ be a measure space. For any numeric function

$$f : X \to \overline{\mathbb{R}}$$

and for any complex valued function

$$f : X \to \mathbb{C}$$

we put

$$N_\infty(f) := \inf_{\substack{N \in \mathscr{A} \\ \mu(N)=0}} \sup_{x \in X \setminus N} |f(x)|.$$

Remark that we do not exclude the possibility to have $N_\infty(f) = +\infty$ for some f. But if $N_\infty(f) < +\infty$, this means that f is *bounded almost everywhere*.

Example 173.
Consider the numeric function

$$f(x) := +\infty \cdot \chi_{\mathbb{Q}}(x) = \begin{cases} +\infty & \text{, if } x \in \mathbb{R} \\ 0 & \text{, elsewhere.} \end{cases}$$

We consider this numeric function on the measure space

$$X = \mathbb{R}, \quad \mathscr{A} = \mathscr{L}^1, \quad \mu = \lambda^1.$$

Since $\lambda^1(\mathbb{R}) = 0$, we have

$$\sup_{x \in \mathbb{R} \setminus \mathbb{Q}} f(x) = 0$$

and thus

$$N_\infty(+\infty \cdot \chi_{\mathbb{Q}}) = 0.$$

The notion of $\mathscr{L}^\infty$-spaces

Definition 174.
Given: A measure space $(X, \mathscr{A}, \mu)$ and a constant $p \geq 1$
we define: The Lebesgue spaces $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ and $\mathscr{L}^\infty_{\mathbb{C}}(X, \mathscr{A}, \mu)$ as:

$$\mathscr{L}^\infty(X, \mathscr{A}, \mu) := \{ f \in \overline{\mathscr{Z}}(X, \mathscr{A}) \ : \ N_\infty(f) < +\infty \}$$

and

$$\mathscr{L}^\infty_{\mathbb{C}}(X, \mathscr{A}, \mu) := \Big\{ f : X \to \mathbb{C} \ : \ \Im f, \Re f \in \mathscr{Z}(X, \mathscr{A}) \\ N_\infty(f) < +\infty \Big\}.$$

Addition and multiplication by scalars in $\mathscr{L}^\infty$-spaces

We introduce now an addition and a multiplication by scalars on $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$. We invite the reader to treat the case $\mathscr{L}^\infty_{\mathbb{C}}(X, \mathscr{A}, \mu)$ in parallel by himself,

We put

- $+ : \mathscr{L}^\infty(X, \mathscr{A}, \mu) \times \mathscr{L}^\infty(X, \mathscr{A}, \mu) \to \mathscr{L}^\infty(X, \mathscr{A}, \mu)$ is given by pointwise addition:

$$(f + g)(x) := f(x) + g(x) \qquad (x \in X).$$

- $\cdot : \mathbb{R} \times \mathscr{L}^\infty(X, \mathscr{A}, \mu) \to \mathscr{L}^\infty(X, \mathscr{A}, \mu)$ is given by pointwise multiplication:

$$(\alpha \cdot f)(x) := \alpha \cdot f(x) \qquad (x \in X).$$

These definitions could be somewhat problematic, since for example it is not clear that

$$f, g \in \mathscr{L}^\infty(X, \mathscr{A}, \mu) \Longrightarrow f + g \in \mathscr{L}^\infty(X, \mathscr{A}, \mu).$$

or that

$$\alpha \in \mathbb{R}, f \in \mathscr{L}^p(X \mathscr{A}, \mu) \Longrightarrow \alpha \cdot f \in \mathscr{L}^p(X \mathscr{A}, \mu).$$

I fact we know that $f + g$ and $\alpha \cdot f$ are measurable functions, but we must check whether or not these functions belong to $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$.

Again, this is easy for $\alpha \cdot f$ and follows from

$$|\alpha \cdot f(x)| = |\alpha| \cdot |f(x)|$$

and form

$$\sup_{x \in X \setminus N} |\alpha \cdot f(x)| = |\alpha| \cdot \sup_{x \in X \setminus N} |f(x)| \qquad \forall N \in \mathscr{A} \text{ with } \mu(N) = 0,$$

i.e.

$$N_\infty(\alpha \cdot f) = |\alpha| \cdot N_\infty(f).$$

The following proposition shows that addition on $\mathscr{L}^\infty$-spaces is well defined, too.

Proposition 175.
We have

$$N_\infty(f + g) \le N_\infty(f) + N_\infty(g)$$

and thus

$$f, g \in \mathscr{L}^\infty(X, \mathscr{A}, \mu) \Longrightarrow f + g \in \mathscr{L}^\infty(X, \mathscr{A}, \mu).$$

Proof. Let $\varepsilon > 0$ (and small) be fixed, and choose the N_1 and $N_2 \in \mathscr{A}$ with

- $\mu(N_k) = 0$, for $k = 1, 2$;
- $\sup_{x \in X \setminus N_1} |f(x)| \le N_\infty(f) + \varepsilon/2$ and $\sup_{x \in X \setminus N_2} |g(x)| \le N_\infty(g) + \varepsilon/2$.

Then, setting $N := N_1 \cup N_2$, we obtain

- $\mu(N) = 0$;
- $$\begin{aligned} N_\infty(f+g) &\leq \sup_{x \in X \setminus N} |f(x)+g(x)| \\ &\leq \sup_{x \in X \setminus N} |f(x)| + \sup_{x \in X \setminus N} |g(x)| \\ &\leq \sup_{x \in X \setminus N_1} |f(x)| + \sup_{x \in X \setminus N_2} |g(x)| \\ &\leq N_\infty(f) + \varepsilon/2 + N_\infty(g) + \varepsilon/2. \end{aligned}$$

Letting $\varepsilon \to 0^+$, we get the claim. □

$\mathscr{L}^p$-spaces are linear spaces for $p \in [1, +\infty]$

Thus we get

Proposition 176. *Let $p \in [1, +\infty]$ be fixed. Then*

$$\mathscr{L}^p(X, \mathscr{A}, \mu) := \left\{ f : X \to \overline{\mathbb{R}} \;:\; f \in \overline{\mathscr{Z}}(X, \mathscr{A}), \quad N_p(f) < +\infty \right\}$$

and

$$\mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu) := \left\{ f : X \to \mathbb{C} \;:\; \Re f, \Im f \in \mathscr{Z}(X, \mathscr{A}), \quad N_p(f) < +\infty \right\}.$$

when equipped with the above defined addition $f + g$ and scalar multiplication $\alpha \cdot f$ are both linear spaces over $\mathbb{R}$ resp. $\mathbb{C}$.

5.2. Normed spaces and convergence

5.2.1. The concept of norm

The notion of norm and of normed space

Definition 177.

Given: A linear space X over $\mathbb{K}$

we say: X a normed space iff:

there exists a *norm on* X, i.e. iff there exists a mapping

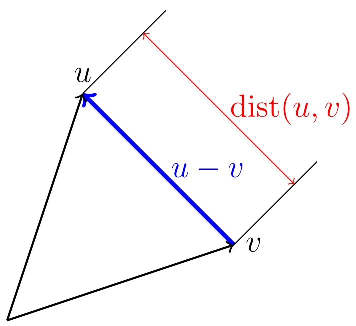
$$\|\cdot\| : X \to \mathbb{R}$$

exhibiting the following properties:

- **strict positivity:** We have
  - $\|u\| \geq 0 \qquad \forall u \in X$ and
  - $\|u\| = 0 \iff u = 0.$
- **homogeneity:** $\|\alpha \cdot u\| = |\alpha| \cdot \|u\|, \forall \alpha \in \mathbb{K}$ and $\forall u \in X$.
- **triangular inequality** $\|u + v\| \leq \|u\| + \|v\|, \forall u, v \in X$.

Remark 178. *In a normed space, it makes sens to speak about the* distance between two points u and v. *By this we mean the value of*

$$\text{dist}(u, v) = \|u - v\|.$$



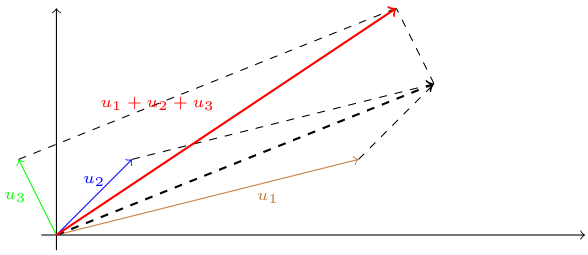
- ***strict positivity:*** *for all* u, v,
  $\text{dist}(u, v) \geq 0$ *and*
  $\text{dist}(u, v) = 0 \iff u = v$
- ***symmetry:*** *for all* u, v,
  $\text{dist}(u, v) = \text{dist}(v, u)$
- ***triangular inequality:*** *for all* u, v, w,
  $\text{dist}(u, v) \leq \text{dist}(u, w) + \text{dist}(w, v)$

Remark 179. *The triangular inequality can be extended to more than two elements. Indeed, one gets*

$$\|(u + v) + w\| \leq \|u + v\| + \|w\| \leq \|u\| + \|v\| + \|w\|$$

and thus

$$\left\| \sum_{k=1}^{n} u_k \right\| \leq \sum_{k=1}^{n} \|u_k\| \qquad (n = 2, 3, 4, \ldots).$$



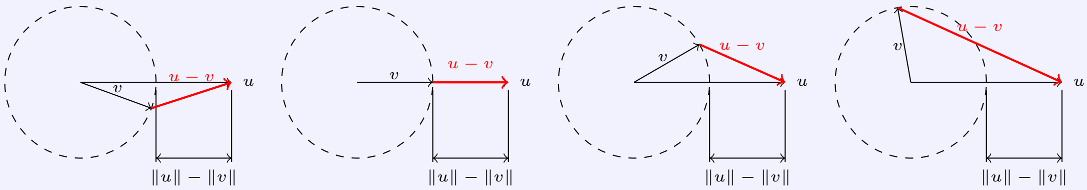
Generalized triangular inequality

The triangular inequality can be written in a generalized version:

Proposition 180.

Suppose that $(X, \|\cdot\|)$ *is a normed space.*
Then we have, $\forall u, v \in X$*,*

$$\Big| \|u - v\| \Big| \le \|u \pm v\| \le \|u\| + \|v\|.$$



Proof. Concerning the second inequality, we may argue as follows:

$$\|u \pm v\| = \|u + (\pm v)\| \le \|u\| + \|\pm v\| = \|u\| + \|v\|.$$

Concerning the first inequality, we first remark that

$$\|u\| = \|(u - v) + v\| \le \|u - v\| + \|v\|$$

so that

$$\|u\| - \|v\| \le \|u - v\|$$

By symmetry, we have

$$\|v\| - \|u\| \le \|v - u\| = \|-(v - u)\| = \|u - v\|.$$

This gives the second inequality, since

$$\Big| \|u\| - \|v\| \Big| \le \|u - v\| \qquad \text{and} \qquad \Big| \|u\| - \underbrace{\|-v\|}_{=\|v\|} \Big| \le \|u + v\|.$$

□

5.2.2. Convergence

We can say what is close to a given point

As yet mentioned, in a normed space we can speak about the distance of two points:

$$\text{dist}(u, v) = \|v - u\|$$

Thus, we have a notion for "a point u to be close to another point v". Thus, we can introduce the concept of *convergence of a sequence*.

The notion of convergence

Definition 181.

Given: A sequence $\{u_n\}_{n=1}^{+\infty}$ in a normed space $(X, \|\cdot\|)$

we say: the sequence $\{u_n\}_{n=1}^{+\infty}$ converges to a point $u \in X$ iff:

$$\lim_{n\to\infty} \|u_n - u\| = 0.$$

i.e. iff for every tolerance $\varepsilon > 0$, there exists a threshold $n_0 = n_0(\varepsilon)$ (depending on the given ε) such that

$$\|u_n - u\| < \varepsilon, \qquad \forall n \geq n_0.$$

We write in this case

$$\lim_{n\to\infty} u_n = x \qquad \text{or} \qquad u_n \to u \quad (\text{as } n \to \infty).$$

Remark 182. *Remark that the limit point u, if it exists, is uniquely determined.*

Indeed, if we assume that

$$\lim_{n\to\infty} u_n = u \qquad \textit{and} \qquad \lim_{n\to\infty} u_n = v$$

we get

$$\begin{aligned} \|u - v\| &= \|(u - u_n) - (v - u_n)\| \\ &\leq \|u - u_n\| + \|v - u_n\| \to 0 \quad (\textit{as } n \to \infty) \end{aligned}$$

so that

$$\|u - v\| = 0.$$

By the strict positivity of the norm, this implies that

$$u = v.$$

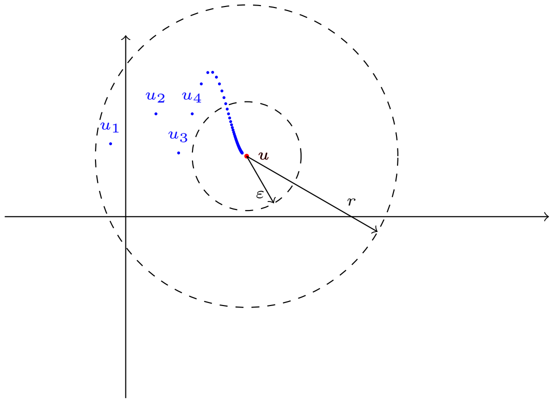
Convergent sequences are bounded

Proposition 183.
Every convergent sequence $\{u_n\}_{n=1}^{+\infty}$ *in a normed space* $(X, \|\cdot\|)$ *is bounded, i.e.*

$$\exists R \geq 0 \text{ such that } \|u_n\| \leq R, \quad \forall n \in \{1, 2, 3, \dots\}.$$

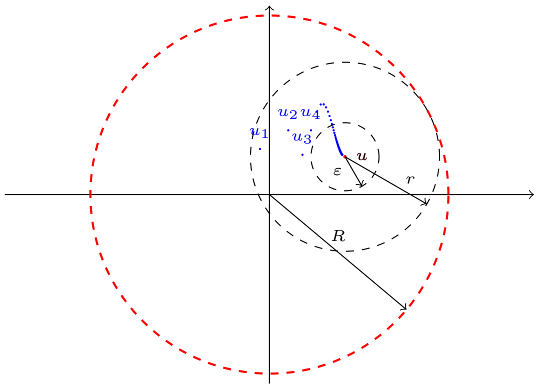
Proof. Indeed

$$\lim_{n\to\infty} \|u_n - u\| = 0 \Longrightarrow \exists r \geq 0 \text{ with } \|u_n - u\| \leq r, \quad \forall n \in \{1, 2, 3, \dots\}.$$



Hence

$$\|u_n\| = \|(u_n - u) + u\| \leq \|u_n - u\| + \|u\| \leq r + \|u\| := R.$$



□

5.2.3. Continuity of the norm and the operations

Continuity of the norm

Proposition 184.

Hyp A normed space $(X, \|\cdot\|)$ containing elements that we denote by u_n and u

Concl The norm

$$\|\cdot\| : X \to \mathbb{R}, \quad u \mapsto \|u\| \quad (\geq 0)$$

is continuous. This means that

$$u_n \to u \quad (\text{as } n \to \infty) \Longrightarrow \|u_n\| \to \|u\| \quad (\text{as } n \to \infty)$$

i.e.

$$\lim_{n\to\infty} u_n = u \Longrightarrow \lim_{n\to\infty} \|u_n\| = \|u\|.$$

Proof. This follows from

$$\Big| \|u_n\| - \|u\| \Big| \leq \|u_n - u\| \to 0 \quad (\text{as } n \to \infty).$$

□

Continuity of the operations

Proposition 185.

Hyp

- $(X, +, \cdot)$ a $\mathbb{K}$-linear space equipped with a norm $\|\cdot\|$
- u_n, u, v_n and v elements of X, and α_n and α elements of $\mathbb{K}$.

Concl

1. The addition

$$+ : X \times X \to X, (u, v) \mapsto u + v$$

is continuous. This means that

$$\left.\begin{array}{ll} u_n \to u & (\text{as } n \to \infty) \\ v_n \to v & (\text{as } n \to \infty) \end{array}\right\} \Longrightarrow u_n + v_n \to u + v \quad (\text{as } n \to \infty).$$

2. *The scalar multiplication*

$$\cdot : \mathbb{K} \times X \to X, (\alpha, u) \mapsto \alpha \cdot u$$

is continuous. This means that

$$\left.\begin{array}{ll} \alpha_n \to \alpha & (\text{as } n \to \infty) \\ u_n \to u & (\text{as } n \to \infty) \end{array}\right\} \Longrightarrow \alpha_n \cdot u_n \to \alpha \cdot u \quad (\text{as } n \to \infty).$$

Proof. The first point follows from

$$\begin{aligned} \|(u_n + v_n) - (u + v)\| &= \|(u_n - u) + (v_n - v)\| \\ &\leq \|u_n - u\| + \|v_n - v\| \to 0 \quad (\text{as } n \to \infty) \end{aligned}$$

The second point follows from

$$\begin{aligned} \|\alpha_n \cdot u_n - \alpha \cdot u\| &= \|(\alpha_n - \alpha)u_n + \alpha(u_n - u)\| \\ &\leq \|(\alpha_n - \alpha)u_n\| + \|\alpha(u_n - u)\| \\ &\leq \underbrace{|\alpha_n - \alpha|}_{\to 0} \cdot \underbrace{\|u_n\|}_{\leq R} + |\alpha| \cdot \underbrace{\|u_n - u\|}_{\to 0}. \end{aligned}$$

Remark that the sequence $\{u_n\}_{n=1}^{+\infty}$, being a convergent sequence, is bounded (by say R). Thus we get the claim when $n \to \infty$. □

5.2.4. The normed space $L^p(X, \mathscr{A},)\mu$

Definition of $\mathscr{L}^p(X, \mathscr{A}, \mu)$

Recall that, for $1 \leq p \leq \infty$,

$$\mathscr{L}^p(X, \mathscr{A}, \mu) := \{f : X \to \overline{\mathbb{R}} \quad ; \quad f \in \overline{\mathscr{Z}}(X, \mathscr{A}), \quad N_p(f) < +\infty\}$$

and

$$\mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu) := \{f : X \to \mathbb{C} \quad : \quad \Im f, \Re f \in \mathscr{Z}(X, \mathscr{A}) \quad \text{and } N_p(f) < +\infty\},$$

where in both cases (the real-valued as well as the complex-valued case) we have

$$N_p(f) = \int_X |f(x)|^p \, d\mu(x) \qquad , \text{for } 1 \leq p < \infty$$

and

$$N_\infty(f) = \inf_{\substack{N \in \mathscr{A} \\ \mu(N) = 0}} \sup_{x \in X \setminus N} |f(x)|.$$

The linear space $\mathscr{L}^p(X, \mathscr{A}, \mu)$

Moreover, we yet know that

Proposition 186.

<u>*Hyp*</u> $1 \leq p \leq +\infty$

<u>*Concl*</u>

1. *Both* $\mathscr{L}^p(X, \mathscr{A}, \mu)$ *and* $\mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$ *are linear spaces.*
   (This is so for $p = \infty$*, too.)*

2. $N_p(\cdot)$ *(inclusively for* $p = \infty$*) is positive, i.e.*

   $$N_p(f) \geq 0, \qquad \forall f \in \mathscr{L}^p(X, \mathscr{A}, \mu) \text{ resp. } f \in \mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu).$$

   However, in general, $N_p(\cdot)$ *is* not *strictly positive.*

3. $N_p(\cdot)$ *is homogeneous, i.e.*

   $$N_p(\alpha \cdot f) = |\alpha| \cdot N_p(f),$$

   $\forall f \in \mathscr{L}^p(X, \mathscr{A}, \mu)$ *resp.* $f \in \mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$ *and* $\forall \alpha \in \mathbb{R}$ *resp.* $\alpha \in \mathbb{C}$.

Remark 187. *Moreover, we yet know that* $N_\infty(\cdot)$ *satisfies the triangular inequality:*

$$N_\infty(f + g) \leq N_\infty(f) + N_\infty(g), \qquad \forall f, g.$$

We say that

$$\boxed{N_\infty(\cdot) \text{ is a semi-norm on } \mathscr{L}^\infty(X, \mathscr{A}, \mu) \text{ resp. } \mathscr{L}^\infty_{complex}(X, \mathscr{A}, \mu)}$$

since $N_\infty(\cdot)$ *satisfies all properties of a norm except the strict positivity.*

We will now show that, for $1 \leq p < \infty$*,* $N_p(\cdot)$ *is a semi-norm, too. We will proceed in two steps:*

1. *the Hölder inequality (a result that is useful by itself)*

2. *the Minkowski inequality (corresponding to the triangular inequality).*

Hölder inequality

An upper bound for a product by a sum of squares

We need a preparatory lemma, a result that is useful by itself.

We intend to generalize the fact that

$$2ab \leq a^2 + b^2 \qquad \forall, a, b \in]0, +\infty[,$$

a result that follows from
$$0 \le (a-b)^2 = a^2 + b^2 - 2ab.$$
Usually, the above result is written in the following from:
$$\boxed{a \cdot b \le \frac{a^2}{2} + \frac{b^2}{2}, \qquad \forall a, b \in]0, +\infty[}.$$

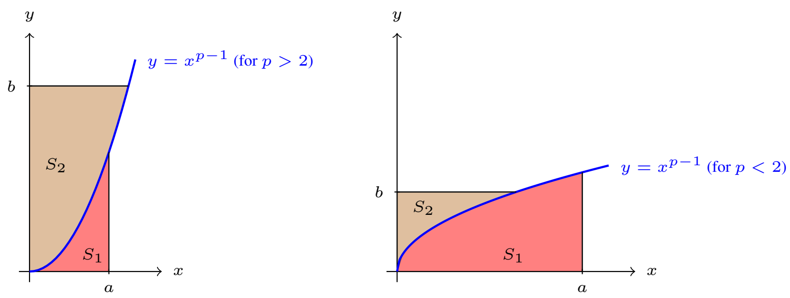
Young's inequality

Proposition 188.

<u>*Hyp*</u> $p, q \in]1, +\infty[$ *a conjugate pair, i.e. suppose that*
$$\frac{1}{p} + \frac{1}{q} = 1$$
(for example $p = q = 2$, or $p = 3$ and $q = 3/2$.)

<u>*Concl*</u>
$$\boxed{a \cdot b \le \frac{a^p}{p} + \frac{b^q}{q}, \qquad \forall a, b \in]0, +\infty[}$$

Proof. For $x > 0$, we consider the curve $y = x^{p-1}$.



We use that fact that
$$a \cdot b \le S_1 + S_2.$$
We can determine the areas S_1 and S_2 in the following way:
$$S_1 = \int_0^a x^{p-1}\, dx = \frac{a^p}{p}$$
and
$$S_2 = \int_0^b y^{\frac{1}{p-1}}\, dy = \int_0^b y^{q-1}\, dy = \frac{b^q}{q}.$$

Remark that

$$\frac{1}{p} = 1 - \frac{1}{q} = \frac{q-1}{q}$$

gives

$$p = \frac{q}{q-1}, \qquad p - 1 = \frac{1}{q-1} \qquad \text{and} \qquad \frac{1}{p-1} = q - 1.$$

Thus we get

$$a \cdot b \leq S_1 + S_2 = \frac{a^p}{p} + \frac{b^q}{q}.$$

□

Hölder's inequality

Proposition 189.

Hyp p *and* q *a conjugate pair with* $p, q \geq 1$. *By this we mean that*

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{if } p, q > 1 \qquad \text{or} \qquad p = 1, q = \infty \qquad \text{or} \qquad p = \infty, q = 1.$$

Concl

1. *If* $f \in \mathscr{L}^p(X, \mathscr{A}, \mu)$ *and* $g \in \mathscr{L}^q(X, \mathscr{A}, \mu)$ *then*

$$f \cdot g \in \mathscr{L}^1(X, \mathscr{A}, \mu)$$

and

$$N_1(f \cdot g) \leq N_p(f) \cdot N_q(g).$$

2. *The same is true if* $f \in \mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$ *and* $g \in \mathscr{L}^q_{\mathbb{C}}(X, \mathscr{A}, \mu)$:

$$f \cdot g \in \mathscr{L}^1_{\mathbb{C}}(X, \mathscr{A}, \mu)$$

and

$$N_1(f \cdot g) \leq N_p(f) \cdot N_q(g).$$

Remark 190. *For* $p > 1$, $q > 1$, *Hölder's inequality means*

$$\boxed{\int_X |f(x) \cdot g(x)| \, d\mu(x) \leq \left[\int_X |f(x)|^p \, d\mu(x)\right]^{1/p} \cdot \left[\int_X |g(x)|^q \, d\mu(x)\right]^{1/q}}$$

For $p = \infty$, $q = 1$, Hölder's inequality means

$$\boxed{\int_X |f(x) \cdot g(x)|\, d\mu(x) \leq \underbrace{\inf_{\substack{N \in \mathscr{A} \\ \mu(N)=0}} \sup_{x \in X \setminus N} |f(x)|}_{=N_\infty(f)} \cdot \int_X |g(x)|\, d\mu(x)}$$

This latter case follows immediately from the fact that for any null-set N, we have

$$\begin{aligned} \int_X |f(x) \cdot g(x)|\, d\mu(x) &\leq \int_{X \setminus N} |f(x) \cdot g(x)|\, d\mu(x) + \underbrace{\infty \cdot \mu(N)}_{=0} \\ &\leq \sup_{x \in X \setminus N} |f(x)| \cdot \int_{X \setminus N} |g(x)|\, d\mu(x) \\ &\leq \sup_{x \in X \setminus N} |f(x)| \cdot \int_X |g(x)|\, d\mu(x) \end{aligned}$$

so that

$$\int_X |f(x) \cdot g(x)|\, d\mu(x) \leq \inf_{\substack{N \in \mathscr{A} \\ \mu(N)=0}} \sup_{x \in X \setminus N} |f(x)| \cdot \int_X |g(x)|\, d\mu(x)$$

We will now prove Hölder's inequality for the former case.

Proof. We apply Young's inequality

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with

$$a := \frac{|f(x)|}{N_p(f)} \qquad \text{and} \qquad b := \frac{|g(x)|}{N_q(f)}$$

and we get in this way

$$\frac{|f(x)| \cdot |g(x)|}{N_p(f) \cdot N_q(g)} \leq \frac{1}{p} \cdot \frac{|f(x)|^p}{N_p(f)^p} + \frac{1}{q} \cdot \frac{|g(x)|^q}{N_q(g)^q}$$

Integrating this inequality, we get

$$\begin{aligned} \frac{1}{N_p(f) \cdot N_q(g)} &\cdot \int_X |f(x) \cdot g(x)|\, d\mu(x) \\ &\leq \frac{1}{p} \cdot \underbrace{\frac{\int_X |f(x)|^p\, d\mu(x)}{N_p(f)^p}}_{=1} + \frac{1}{q} \cdot \underbrace{\frac{\int_X |g(x)|^q\, d\mu(x)}{N_q(g)^q}}_{=1} \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

and this gives the claim.

Remark than when

$$N_p(f) = 0 \qquad \text{or} \qquad N_q(g) = 0$$

Hölder's inequality holds since $f \cdot g = 0$ μ-almost everywhere. In fact, we even have equality: $0 = N_1(f \cdot g) = N_p(f) \cdot N_q(g)$. □

The Minkowsky inequality

Minkowsky inequality

Proposition 191.
Suppose that $1 \le p \le \infty$.
Then $N_p(\cdot)$ *satisfies the triangular inequality, i.e.,* $\forall f, g \in \mathscr{L}^p(X, \mathscr{A}, \mu)$ *resp.* $\forall f, g \in \mathscr{L}^p(X, \mathscr{A}, \mu)$*, we have*

$$N_p(f+g) \le N_p(f) + N_p(g).$$

For $p < \infty$*, this means that*

$$\left[\int_X |f(x)+g(x)|^p \, d\mu(x)\right]^{1/p} \le \left[\int_X |f(x)|^p \, d\mu(x)\right]^{1/p} + \left[\int_X |g(x)|^p \, d\mu(x)\right]^{1/p}.$$

Remark 192. *Thus,* $N_p(\cdot)$ *is a semi-norm for* $1 \le p \le \infty$.

Remark 193. *For* $p = \infty$*, we have yet established the Minkowsky inequality.*
For $p = 1$*, the claim can be obtained by integrating the relation*

$$|f(x)+g(x)| \le |f(x)| + |g(x)|.$$

Thus, we give now the proof for $1 < p < \infty$*. Thereby we put*

$$q := \frac{p}{p-1}$$

and we remark that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We have, since $(p-1)q = p$,

$$\begin{aligned} N_p(f+g)^p &= \int_X |f(x)+g(x)|^p \, d\mu(x) \\ &\le \int_X (|f|+|g|)\, |f+g|^{p-1} \, d\mu \\ &= \int_X |f| \underbrace{|f+g|^{p-1}}_{\in \mathscr{L}^q} \, d\mu + \int_X |g| \underbrace{|f+g|^{p-1}}_{\in \mathscr{L}^q} \, d\mu \end{aligned}$$

Thus we get, by Hölder inequality,

$$\begin{aligned} N_p(f+g)^p &\le \int_X \underbrace{|f|}_{\in \mathscr{L}^p} \underbrace{|f+g|^{p-1}}_{\in \mathscr{L}^q} \, d\mu + \int_X \underbrace{|g|}_{\in \mathscr{L}^p} \underbrace{|f+g|^{p-1}}_{\in \mathscr{L}^q} \, d\mu \\ &\le N_p(f) \cdot \underbrace{N_q(|f+g|^{p-1})}_{=N_p(f+g)^{p-1}} + N_p(g) \cdot \underbrace{N_q(|f+g|^{p-1})}_{=N_p(f+g)^{p-1}} \\ &= \Big(N_p(f) + N_p(g)\Big) \cdot N_p(f+g)^{p-1} \end{aligned}$$

i.e.

$$N_p(f+g) \le N_p(f) + N_p(g).$$

□

In general $N_p(\cdot)$ is only a semi-norm, not a norm!

In general $N_p(f)$ is *not* a norm. However, we have

$$N_p(f) = 0 \Longleftrightarrow f = 0 \ \mu\text{-a.e.}$$

How to get a norm ...

Thus, we collect all "very small functions" in a set M:

$$\begin{aligned} M &:= \{f \in \mathscr{L}^p(X, \mathscr{A}, \mu) \ : \ N_p(f) = 0\} \\ &= \{f \in \mathscr{L}^p(X, \mathscr{A}, \mu) \ : \ f = 0 \ \mu\text{-a.e.}\} \end{aligned}$$

resp.

$$\begin{aligned} M &:= \{f \in \mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu) \ : \ N_p(f) = 0\} \\ &= \{f \in \mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu) \ : \ f = 0 \ \mu\text{-a.e.}\} \end{aligned}$$

Remark that M is a linear subspace.

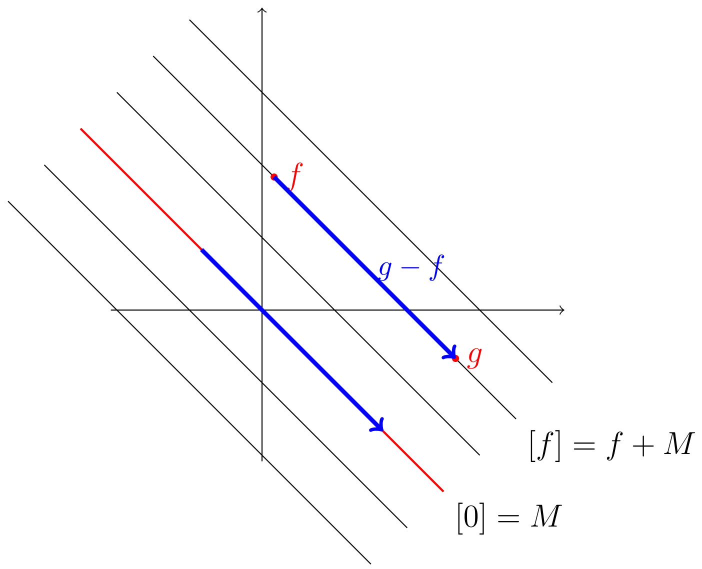
Then, we introduce an equivalence relation via

$$f \sim g :\Longleftrightarrow f - g \in M \Longleftrightarrow f = g \ \mu\text{-a.e.}$$

Remark that this relation has the following properties (corresponding to the fact to be an equivalence relation)

- **reflexivity:** $f \sim f, \forall f$
- **symmetry:** $f \sim g \Longrightarrow g \sim f, \forall g, f$
- **transitivity:**

$$\left.\begin{matrix} f \sim g \\ g \sim h \end{matrix}\right\} \Longrightarrow f \sim h, \qquad \forall f, g, h.$$



We consider now the quotient space

$$L^p(X, \mathscr{A}, \mu) := \mathscr{L}^p(X, \mathscr{A}, \mu)|_M := \{[f] := f + M \ : \ f \in \mathscr{L}^p(X, \mathscr{A}, \mu)\}$$

resp.

$$L^p_{\mathbb{C}}(X, \mathscr{A}, \mu) := \mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu)|_M := \{[f] := f + M \ : \ f \in \mathscr{L}^p_{\mathbb{C}}(X, \mathscr{A}, \mu)\}$$

equipped with

1. the addition $[f] + [g] := [f + g]$;
2. the scalar multiplication $\alpha \cdot [f] := [\alpha \cdot f]$;
3. the norm $\|[f]\|_p := N_p(f)$.

The notion of L^p-spaces

Definition 194.
We call the so obtained spaces Lebesgue spaces $L^p(X, \mathscr{A}, \mu)$, resp. $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$ (with $1 \leq p \leq \infty$).

We must now make some remarks:

- the first group of remarks concerns the fact that the above introduced addition, scalar multiplication and norm are well-defined; by this we mean that their definition does not depend on the choice of the representative elements.
- the second group of remarks concerns the validity of Hölder and Minkowsky inequalities.

The addition $[f] + [g]$ is well defined

Remark 195. *The addition $[f] + [g] := [f + g]$ in $L^p(X, \mathscr{A}, \mu)$ (resp. $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$) is well-defined. By this we mean that the answer of this addition does not depend on the choice of the representatives.*

Indeed, let us assume that

$$f_1 = f_2 \quad \mu\text{-a.e.} \qquad \text{and} \qquad g_1 = g_2 \quad \mu\text{-a.e.}$$

so that $[f_1] = [f_2]$ and $[g_1] = [g_2]$. Then

$$f_1 + g_1 = f_2 + g_2 \quad \mu\text{-a.e.}$$

so that $[f_1 + g_1] = [f_2 + g_2]$.

Scalar multiplication $\alpha \cdot [f]$ is well-defined

Remark 196. *The scalar multiplication $\alpha \cdot [f] := [\alpha \cdot f]$ in $L^p(X, \mathscr{A}, \mu)$ (resp. $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$) is well-defined. By this we mean that the answer of this product does not depend on the choice of the representative.*

This is so by a similar argument as the one we have just used for the addition:

$$f_1 = f_2 \quad \mu\text{-a.e.} \Longrightarrow \alpha \cdot f_1 = \alpha \cdot f_2 \quad \mu\text{-a.e.}$$

$\|\cdot\|_p$ is well-defined

Remark 197. *$\|.\|_p$ is well-defined. By this we mean that the result does not depend on the representative, i.e.,*

$$f_1 = f_1 \quad \mu\text{-a.e.} \Longrightarrow N_p(f_1) = N_p(f_2).$$

Indeed

$$\begin{aligned} N_p(f_1) &= N_p(f_1 - f_2 + f_2) \leq \underbrace{N_p(f_1 - f_2)}_{=0} + N_p(f_2) = N_p(f_2) \\ N_p(f_2) &= N_p(f_2 - f_1 + f_1) \leq \underbrace{N_p(f_2 - f_1)}_{=0} + N_p(f_1) = N_p(f_1) \end{aligned}$$

so that $N_p(f_1) = N_p(f_2)$.

The L^p-spaces are normed spaces

Proposition 198.
The Lebesgue space $L^p(X, \mathscr{A}, \mu)$ resp. $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$ with $1 \leq p \leq \infty$ is a normed space. The corresponding norm is

- *For $1 \leq p < +\infty$:*

$$\|[f]\|_p := N_p(f) := \left[\int_X |f(x)|^p \, d\mu(x)\right]^{1/p}.$$

- *For $p = +\infty$:*

$$\|[f]\|_\infty = \inf_{\substack{N \in \mathscr{A} \\ \mu(N)=0}} \sup_{x \in X \setminus N} |f(x)|.$$

Proof. We must only show, that $\| \cdot \|_p$ is a norm. In doing this, we first recall that we have yet established that $N_p(\cdot)$ is a semi-norm.

1. $\| \cdot \|_p$ is strictly positive: indeed,

$$\forall f, \quad \|[f]\| := N_p(f) \geq 0.$$

   Moreover

$$\|[f]\|_p = N_p(f) = 0 \Longrightarrow f = 0 \ \mu\text{-a.e.} \Longrightarrow [f] = [0].$$

2. $\| \cdot \|$ is homogeneous:

$$\|\alpha \cdot [f]\|_p = N_p(\alpha \cdot f) = |\alpha| \cdot N_p(f) = \alpha \cdot \|[f]\|.$$

3. $\| \cdot \|$ satisfies the triangular inequality:

$$\|[f] + [g]\| \quad = \quad \|[f + g]\| = N_p(f + g) \leq N_p(f) + N_p(g) = \|f\|_p + \|g\|_p.$$

□

The Hölder and the Minkowsky inequalities in L^p-spaces

Remark 199. *The Hölder and the Minkowsky inequalities remainvalid in $L^p(X, \mathscr{A}, \mu)$ resp. in $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$, for $1 \leq p \leq \infty$.*

- *We have, if $1/p + 1/q = 1$ or $p = \infty, q = 1$ or $p = 1, q = \infty$,*

$$\left.\begin{array}{l} [f] \in L^p \\ [g] \in L^p \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} [f] \cdot [g] \in L^1 \\ \|[f] \cdot [g]\|_p \leq \|[f]\|_p \cdot \|[g]\|_p, \end{array}\right.$$

  where L^p stand for $L^p(X, \mathscr{A}, \mu)$ resp. $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$.

- *We have, for $1 \leq p \leq \infty$,*

$$[f], [g] \in L^p \Longrightarrow \|[f] + [g]\|_p \leq \|[f]\|_p + \|[g]\|_p,$$

  where again L^p stand for $L^p(X, \mathscr{A}, \mu)$ resp. $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$.

Convergence in L^p-spaces

Remark 200. *It make sense to speak of* the convergence of a sequence $\{[f_n]\}_{n=1}^{\infty}$ *in* $L^p(X, \mathscr{A}, \mu)$ *resp.* $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$ *since*

$$\left.\begin{array}{l} f_n = g_n \ \mu\text{-a.e.} \\ \lim_{n\to\infty} f_n = f \\ \lim_{n\to\infty} g_n = g \end{array}\right\} \Longrightarrow f = g \ \mu\text{-a.e.}$$

so that

$$\lim_{n\to\infty} [f_n] = [\lim_{n\to\infty} f_n]$$

(if and only if $\lim_{n\to\infty} f_n$ *exists).*

Identification of f and $[f]$

It is usual to identify

$$f \quad \text{with} \quad [f]$$

for elements in $L^p(X, \mathscr{A}, \mu)$ resp. $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$ $(1 \leq p \leq \infty)$ by saying:

"We identify μ*-a.e. equal functions."*

We will do this from now on and *write* f *instead of* $[f]$.

5.2.5. Cauchy sequences

How to establish convergence: the brute method

If one has to establish the convergence of a given sequence $\{u_n\}_{n=1}^{+\infty}$, one has

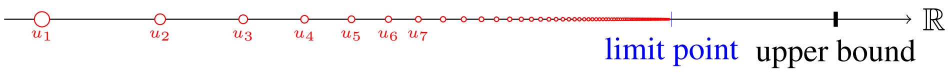
1. to find (by any clever argument, by "illumination" or simple by a "lucky punch") the limit u and
2. to show that
$$\lim_{n\to\infty} \|u_n - u\| = 0.$$

How to establish convergence: better methods exist

But we know from the calculus in $\mathbb{R}$ that we can establish the convergence of a sequence without explicitly knowing its limit. As an example, *every monotone non-decreasing and bounded sequence in* $\mathbb{R}$ *is convergent*. As such an example let us mention

$$u_n := \left(1 + \frac{1}{n}\right)^n$$

with $\lim_{n\to\infty} u_n = e$.



Establishing "monotonicity and "boundedness" is often easier than to find the limit point.

The "inner" behavior of convergent sequences

Let us now consider a convergent sequence $\{u_n\}_{n=1}^{+\infty}$ in a normed space $(X, \|\cdot\|)$:

$$\lim_{n\to\infty} u_n = u \quad \text{, for some } u \in X.$$

Thus, for any given tolerance $\varepsilon > 0$, there exists a threshold $n_0 = n_o(\varepsilon)$ such that

$$\|u_n - u\| < \frac{\varepsilon}{2} \qquad \text{, as soon as } n \geq n_0.$$

Thus, as soon as $n \geq n_0$ and $m \geq n_0$, on gets

$$\|u_n - u_m\| = \|(u_n - u) - (u_m - u)\| \leq \underbrace{\|u_n - u\|}_{<\varepsilon/2} + \underbrace{\|u_m - u\|}_{<\varepsilon} < \varepsilon$$

Metaphorically speaking, this means that in a convergent sequence $\{u_n\}_{n=1}^{+\infty}$ *the elements must move closer each other as soon as the numbering n of the sequence elements is getting larger and larger*:

$$\forall n, m \geq n_0(\varepsilon), \qquad \|u_n - u_m\| < \varepsilon.$$

Remark that this property does not need the knowledge of a limit point, it is an "inner" property of the sequence.

This phenomenon will play a central role in what follows, so we fix it in a definition.

The notion of Cauchy sequence

Definition 201.

<u>Given:</u> A sequence $\{u_n\}_{n=1}^{+\infty}$ in a normed space $(X, \|\cdot\|)$

we say: <u>the sequence $\{u_n\}_{n=1}^{+\infty}$ is a Cauchy sequence</u> iff:

for any given tolerance $\varepsilon > 0$, there exists a threshold $n_0 = n_0(\varepsilon)$ such that

$$\|u_n - u_m\| < \varepsilon \qquad \forall n, m \geq n_0$$

(i.e. the elements in the tail are close each other).

As we have seen above, the following result holds

Proposition 202.
Every convergent sequence $\{u_n\}_{n=1}^{+\infty}$ in a normed space is a Cauchy sequence.

Cauchy $\Longleftrightarrow$ convergent?

Proposition 203.
Any Cauchy sequence in $\mathbb{R}$ (or in $\mathbb{C}$) is convergent.
This is no longer true in $\mathbb{Q}$.

Thus, unfortunately, we cannot establish the convergence of a sequence by showing that the sequence is a Cauchy sequence, unless we are in a "nice" space (like for example $\mathbb{R}$) where the convergence of any Cauchy sequence has be established.

"nice" space X	"not nice" space X
any sequence $\{u_n\}_{n=1}^{+\infty}$ in this space is	
Cauchy $\Longleftrightarrow$ convergent	Cauchy $\Longleftarrow$ convergent
examples of such spaces are	
$\mathbb{R}$ equipped with the norm $\lvert\cdot\rvert$ (absolute value)	$\mathbb{Q}$ equipped with the norm $\lvert\cdot\rvert$ (absolute value)

The following example shows that normed space that are quite "natural" may not be nice in the above sens.

A function space with a non-convergent Cauchy sequence

Example 204.
Let us consider the space

$$X := C[-1,1] := \{u : [-1,1] \to \mathbb{R} \ : \ u \text{ is continuous on [-1,1]}\}.$$

We equip this space by the norm

$$\|u\|_1 := \int_{-1}^{1} |u(x)|\, dx.$$

Remark that $\|\cdot\|_1$ is a norm. Indeed

• $\|\cdot\|$ **is strictly positive:**
This follows form $\|u\|_1 \geq 0$, $\forall u \in C[-1,1]$ and from

$$\|u\|_1 = \int_{-1}^{1} |u(x)|\, dx = 0 \Longrightarrow u(x) \equiv 0 \ (x \in [-1,1]).$$

• $\|\cdot\|$ **is homogeneous:**

For $\alpha \in \mathbb{R}$ and $u \in C[-1,1]$ we have

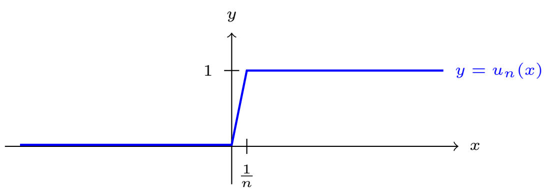
$$\begin{aligned}\|\alpha \cdot u\|_1 &= \int_{-1}^{1} |\alpha \cdot u(x)|\, dx = |\alpha| \cdot \int_{-1}^{1} |u(x)\, dx \\ &= |\alpha| \cdot \|u\|_1.\end{aligned}$$

• $\|\cdot\|$ **satisfies the triangular inequality:**
For $u, v \in C[-1,1]$, we have

$$\begin{aligned}\|u+v\|_1 &= \int_{-1}^{1} \underbrace{|u(x)+v(x)|}_{|u(x)|+|v(x)|}\, dx \\ &\leq \int_{-1}^{1} |u(x)|\, dx + \int_{-1}^{1} |v(x)|\, dx = \|u\|_1 + \|v\|_1.\end{aligned}$$

We consider now the sequence of functions $\{u_n\}_{n=1}^{+\infty}$ in $C[-1,1]$ given by

$$u_n(x) := \begin{cases} 0 & \text{, if } x \leq 0 \\ n \cdot x & \text{, if } 0 \leq x \leq \frac{1}{n} \\ 1 & \text{, if } x \geq \frac{1}{n}. \end{cases}$$

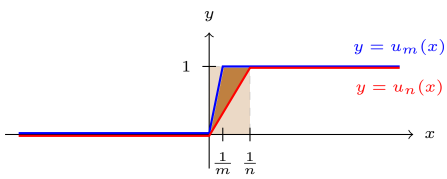


This sequence is Cauchy in $(C[-1,1], \|\cdot\|_1)$, since we have, for $n \leq m$

$$\|u_n - u_m\|_1 = \int_0^{1/n} |u_n(x) - u_m(x)|\, dx \leq \frac{1}{n},$$

and thus

$$\|u_n - u_m\|_1 \leq \varepsilon \qquad \text{, as soon as } n, m \geq \frac{1}{\varepsilon}.$$



Nevertheless, this Cauchy sequence $\{u_n\}_{n=1}^{+\infty}$ does not converge: there exists no limit $u \in C[-1,1]$.

In order to show that, suppose on the contrary that such a limit u exists:

$$\lim_{n\to\infty} \|u_n - u\|_1 = 0.$$

We will show that such a limit u must satisfy

$$u(x) = \begin{cases} 0 & \text{, for } x < 0 \\ 1 & \text{, for } x > 0; \end{cases}$$

But no continuous function of this kind exists, and thus, our Cauchy sequence does not converge.

We will only show that $u(x) = 1$, for $x > 0$, and we leave it to the reader, to verify that $u(x) = 0$ for $x < 0$.

Suppose on the contrary that, for some $\xi > 0$, $u(\xi) \neq 1$. By the assumed continuity of the limit u, we have

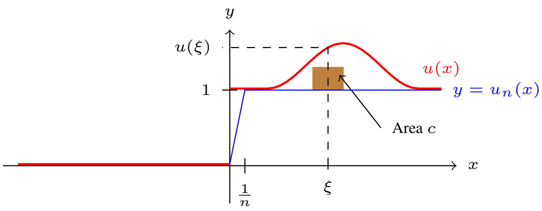
$$u(x) \neq 1 \qquad \text{, for } x \text{ "close" to } \xi.$$

Thus, for n large enough,

$$\|u_n - u\|_1 \geq c > 0,$$

where c is some constant.

The following figure holds for n large enough:



What can be done when the given space has a lot of holes?

Remark 205. *The space $(C[-1,1], \|\cdot\|_1)$ has the same "problem" as the space $(\mathbb{Q}, |\cdot|)$: both seems to have a lot of "holes".*

Two strategies are now possible:

1. *one "fills up" the holes, as it has be done for $(\mathbb{Q}, |\cdot|)$, and one gets a larger space where every Cauchy sequence converges.*

   This can be done for any normed space.

2. *one replaces the given norm $\|\cdot\|_1$ by another norm $\|\cdot\|_2$, so that all Cauchy sequences are convergent; this implies, that it must be more difficult to be a Cauchy sequence with respect to the new norm $\|\cdot\|_2$ than with respect to the starting norm $\|\cdot\|_1$.*

   This process is impossible in finite dimensional spaces. In infinite dimensional spaces, one may be successful, but there is no generic way that helps us when we are looking for the new norm $\|\cdot\|_2$.

The notion of complete space and of completion of a space

Definition 206.
A normed space $(X, \|\cdot\|)$ is complete if any Cauchy sequence is a convergent sequence.

Remark 207. *As yet mentioned above, if a normed space $(X, \|\cdot\|_X)$ is not complete, on can enlarge this space to a larger, complete space $(Y, \|\cdot\|_Y)$ with*

- *$X \subset Y$;*
- *the addition and scalar multiplication, when done in X gives the same result as when done in Y (i.e., operations are preserved).*
- *for all $u \in X$ we have*

$$\|u\|_X = \|u\|_Y \qquad \text{norms are preserved.}$$

This process is called completion. It works for all spaces on the model of the completion of $\mathbb{Q}$.

Remark 208. *If an normed space is not complete, beside technical difficulties one may be confronted to a "counter-intuitive" world.*
Let us give a simple example!

Not complete spaces: a counter-intuitive world

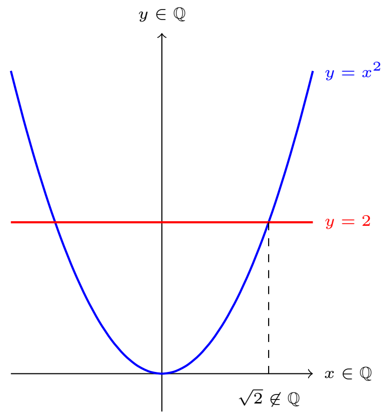
Example 209.
On the non-complete normed space $(\mathbb{Q}, |\cdot|)$, where $|\cdot|$ is the absolute value, one may speak about limits and derivatives of functions.
As an example, the following functions are well-defined, continuous functions having continuous derivatives:

$$\begin{aligned} f : \mathbb{Q} \to \mathbb{Q} &\quad : \quad x \mapsto f(x) := x^2 \qquad \text{with } f'(x) = 2x \\ g : \mathbb{Q} \to \mathbb{Q} &\quad : \quad x \mapsto g(x) := 2 \qquad \text{with } g'(x) = 0. \end{aligned}$$

This is so since, for example,

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

But now, the curves $y = f(x)$ and $y = g(x)$ no longer intersect, despite the following graph:



Complete normed spaces will play a major role in what follows: this will be the topic of the next chapter!

We end this chapter with a remark about normed spaces viewed as topological spaces.

5.3. Normed spaces as topological spaces

Normed spaces are topological spaces

As yet mentioned, in a normed space $(X, \|\cdot\|)$ we have a notion of distance given by

$$\mathrm{dist}(u, v) := \|v - u\|, \qquad \text{for } u, v \in X.$$

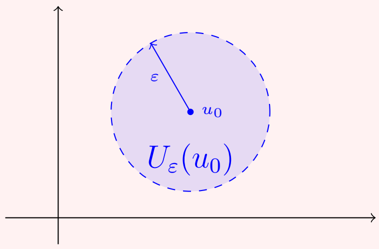
The notion of convergence was based on this notion of distance.

Definition 210.
Given:

- $(X, \|\cdot\|)$ a normed space
- $u_0 \in X$ and $\varepsilon > 0$ kept fixed

we define: an ε-neighborhood of u_0 as:

$$U_\varepsilon(u_0) := \{u \in X \ : \ \|u - u_0\| < \varepsilon\}$$

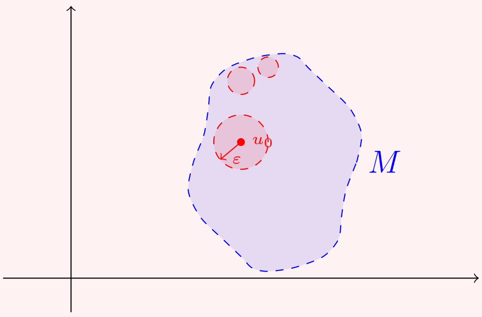


Open and closed sets

Definition 211.

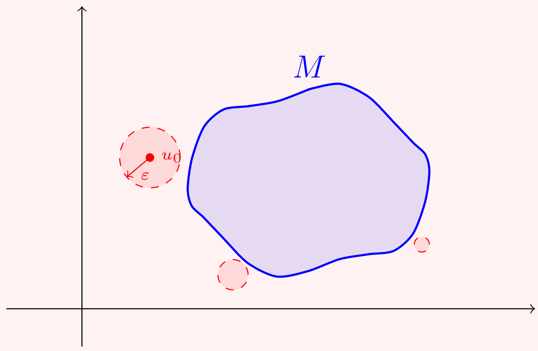
1. Given: A set M in a normed space $(X, \|\cdot\|)$
   we say: M is open iff:

$$\forall u_0 \in M \quad \exists \varepsilon > 0 \text{ such that } U_\varepsilon(u_0) \subset M.$$



the choice of $\varepsilon > 0$ depends on the given point u_0

2. Given: A set M in a normed space $(X, \|\cdot\|)$
we say: M is closed iff:
its complement $\complement M$ is open.



3. By an open neighborhood of u_0 we mean an open set containing u_0.

Another formulation for a set to be closed

Proposition 212.
Let $M \subset X$ be a subset in a normed space $(X, \|\cdot\|)$.
Then the following statements are equivalent:

- *M is closed;*
- *whenever $u_n \to u$ with $u_n \in M$ (for all n), we have $u \in M$, i.e. every limit point u of a sequence $\{u_n\}_{n=1}^{+\infty}$ in M belongs to M.*

Proof. **(I): We show that, if M is closed, then every limit point u of a convergent sequence $\{u_n\}_{n=1}^{+\infty}$ in M belongs to M.**

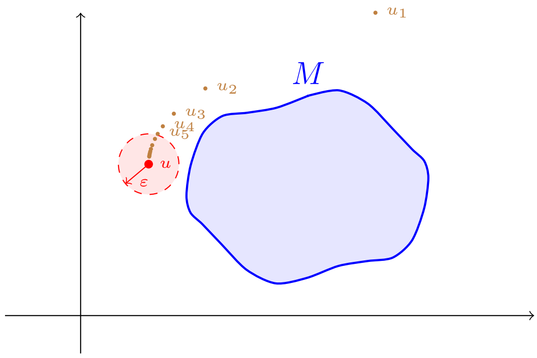
So let $\{u_n\}_{n=1}^{+\infty}$ be a sequence in M converging to some point $u \in X$. We must show that $u \in M$.

In order to show this, assume on the contrary that this is not the case and that

$$u \in \complement M.$$

But M is closed, so its complement $\complement M$ is open. Thus, $\exists \varepsilon > 0$ such that

$$U_\varepsilon(u) \subset \complement M.$$



Now,

$$\lim_{n\to\infty} \|u_n - u\| = 0$$

implies that

$$\|u_n - u\| < \varepsilon \qquad \text{as soon as } n \text{ is large enough, say } n \geq n_0$$

so that

$$u_n \in U_\varepsilon(u) \qquad \text{, for } n \geq n_0.$$

But this would mean that

$$u_n \notin M \qquad \text{, for } n \geq n_0,$$

a contradiction to our hypothesis.

(II): We show that, if the limit point u of any convergent sequence $\{u_n\}_{n=1}^{+\infty}$ in M belongs to M, too, then M is closed.

In order to prove this, we assume on the contrary that M is not closed, and we construct then a convergent sequence $\{u_n\}_{n=1}^{+\infty}$ in M whose limit point u does not belong to M.

Since M is assumed to be not-closed, there exists some $u \in \complement M$ such that, $\forall \varepsilon > 0$,

$$U_\varepsilon(u) \cap M \neq \varnothing.$$

One chooses now, for $n = 1, 2, \ldots$, an element

$$u_n \in U_{\frac{1}{n}}(u) \cap M.$$

Clearly

$$\lim_{n\to\infty} u_n = u$$

with $u_n \in M$ (for all n) and $u \notin M$. This is a contradiction!

So we are done! □

Closed balls are closed

Example 213.

In a norm space $(X, \|\cdot\|)$, one can consider *closed balls with center u_0 and radius $r > 0$*:

$$B_r(u_0) := \{u \in X \ : \ \|u - u_0\| \leq r\}.$$

Any such closed ball is closed.

Indeed, if $\{u_n\}_{n=1}^{+\infty}$ is some sequence in $B_r(u_0)$ converging to some point $\bar{u}$, then

$$\|u_n - u_0\| \leq r \qquad \forall n$$

implies, by continuity of the norm,

$$\|\bar{u} - u_0\| \leq r, \qquad \text{so that } \bar{u} \in M.$$

Thus $B_r(u_0)$ is closed.

Open balls are open

Example 214.

In a norm space $(X, \|\cdot\|)$, all ε-neighborhoods

$$U_\varepsilon(u_0) := \{u \in X \ : \ \|u - u_0\| < \varepsilon\}.$$

are open: thus we may speak of *open balls with center* u_0 *and radius* $\varepsilon > 0$.

This follows from the fact that, for every $u \in U_\varepsilon(u_0)$, we have

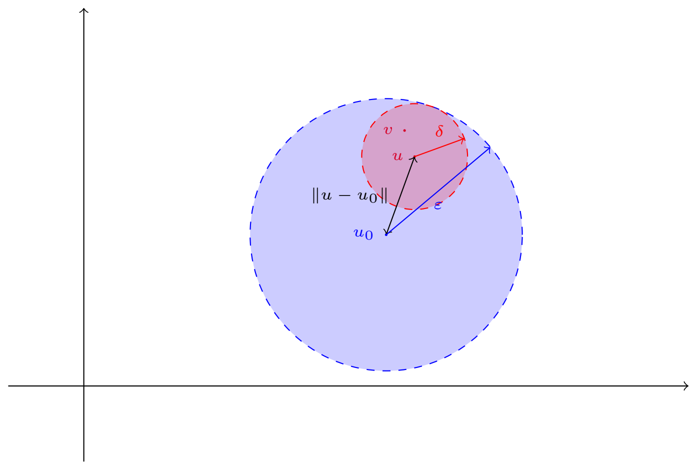
$$U_\delta(u) \subset U_\varepsilon(u_0),$$

if $\delta := \varepsilon - \|u - u_0\|$.

Indeed, $\forall v \in U_\delta(u)$, we have

$$\|v - u_0\| \leq \|v - u\| + \|u - u_0\| < \delta + \|u - u_0\| = \varepsilon,$$

so that $v \in U_\varepsilon(u_0)$.



6

Banach spaces

6.1. Definition of Banach spaces

The notion of Banach space

Definition 215.

Given: A normed space $(X, \|\cdot\|)$

we say: $(X, \|\cdot\|)$ is a Banach space (or a B-space) iff:
$(X, \|\cdot\|)$ is complete, i.e iff every Cauchy sequence $\{u_n\}_{n=1}^{+\infty}$ in X is convergent.

Remark 216. *The space $(\mathbb{Q}, |\cdot|)$, where $|\cdot|$ is the absolute value, is not complete, and hence not a Banach space.*

The space $(\mathbb{R}, |\cdot|)$, where $|\cdot|$ is the absolute value, is complete, and thus a Banach space.

Remark 217. *The above considered space $(C[-1,1], \|\cdot\|_1)$ where*

$$\|u\|_1 := \int_{-1}^{1} |u(x)|\, dx$$

is not *a Banach space.*

As for $\mathbb{R}$, this space may be embedded in a larger space $L^1(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda^1)$ that will be a Banach space (see below).

6.2. Examples of Banach spaced

Example 218.

We equip the linear space $(\mathbb{K}^N, +, \cdot)$, for $N = 1, 2, 3 \ldots$, with

$$\|x\|_\infty := \max_{1 \le j \le N} |\xi_j|, \qquad \text{, where } x = (\xi_1, \ldots, \xi_N).$$

Remark that $\|\cdot\|_\infty$ is a norm on $\mathbb{K}^N$. Indeed

- **Strict positivity:** We have $\|x\| \ge 0$, $\forall x \in \mathbb{K}^N$. Moreover

$$\begin{aligned} \|x\|_\infty = \max_{1 \le j \le N} |\xi_j| = 0 \quad &\iff \quad \xi_j = 0 \text{ for } j = 1, 2, \ldots, N \\ &\iff \quad x = 0. \end{aligned}$$

- **Homogeneity:** We have

$$\|\alpha \cdot x\|_\infty = \max_{1 \le j \le N} \underbrace{|\alpha \cdot \xi_j|}_{=|\alpha| \cdot |\xi_j|} = |\alpha| \cdot \max_{1 \le j \le N} |\xi_j| = |\alpha| \cdot \|x\|_\infty.$$

- **Triangular inequality:** We have

$$\begin{aligned}\|x+y\|_\infty &= \max_{1\le j\le N} \underbrace{|\xi_j+\eta_j|}_{\le|\xi_j|+|\eta_j|} \\ &\le \max_{1\le j\le N}|\xi_j| + \max_{1\le j\le N}|\eta_j| = \|x\|_\infty + \|y\|_\infty.\end{aligned}$$

We show now that

$$\boxed{(\mathbb{K}^N, \|\cdot\|_\infty) \text{ is a Banach space.}}$$

In order to show this, we must show that any given Cauchy sequence in $\mathbb{K}^N$, say

$$\{x_n\}_{n=1}^{+\infty} \qquad (\text{with } x_n := (\xi_1^{(n)}, \dots, \xi_N^{(n)})),$$

is convergent. Thereby we recall that being a Cauchy sequence means

$$\forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon) \text{ s.t.}$$

$$\|x_n - x_m\|_\infty < \varepsilon \text{ as soon as } n, m \ge n_0.$$

Remark that, for $k = 1, 2, \dots, N$

$$|\xi_k^{(n)} - \xi_k^{(m)}| \le max_{1\le j\le N}|\xi_j^{(n)} - \xi_j^{(m)}| = \|x_n - x_m\|_\infty.$$

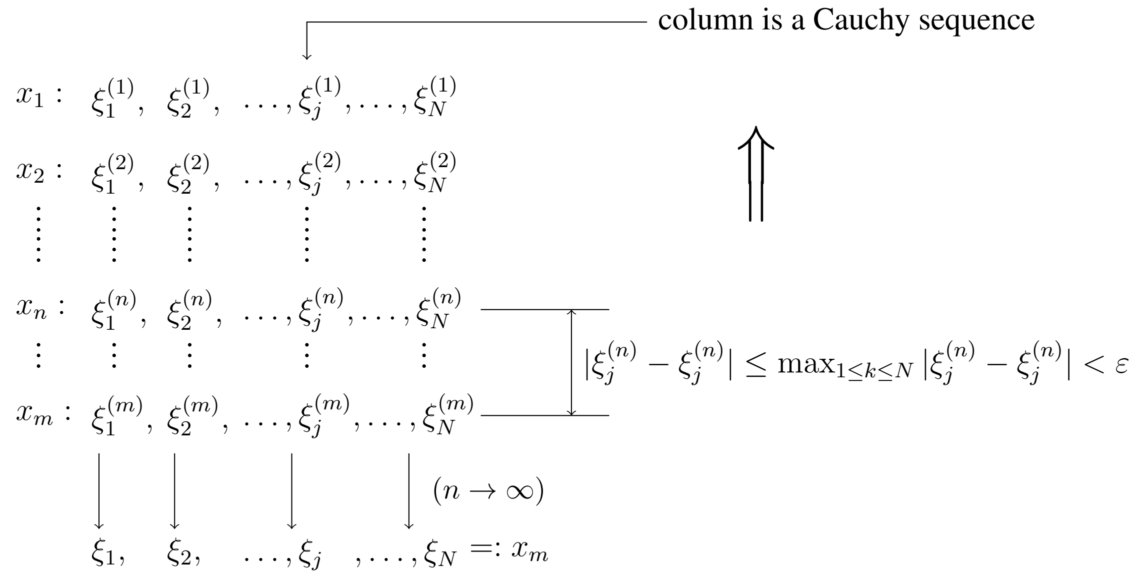
Thus, if the given sequence $\{x_n\}_{n=1}^{+\infty}$ is Cauchy,

$$\text{every sequence } \left\{\xi_k^{(n)}\right\} \text{ in } \mathbb{K} \text{ is Cauchy } (k = 1, 2, \dots, N)$$

and thus convergent:

$$\exists \lim_{n\to\infty} \xi_k^{(n)} =: \xi_k \in \mathbb{K}, \qquad \text{for } k = 1, 2, \dots, N.$$

This means that every component is convergent.



Put now

$$x = (\xi_1, \dots, \xi_N).$$

Then

$$\lim_{n\to\infty} x_n = x$$

since

$$\|x_n - x\|_\infty = \max_{1\le j\le N} |\xi_j^{(n)} - \xi_j| \le \sum_{j=1}^{N} \underbrace{|\xi_j^{(n)} - \xi_j|}_{\to 0 \text{ as } n\to\infty} \to 0 \text{ as } n \to \infty$$

Thus for example $\mathbb{R}^N$ is a Banach space, when equipped with the norm

$$\|x\|_\infty = \max_{1\le j\le N} |\xi_j|, \qquad \text{, if } x = (\xi_1, \dots, \xi_N).$$

This norm is less common, than the usual *Euclidean norm*

$$\|x\|_2 := \sqrt{\sum_{j=1}^{N} \xi^2}.$$

The questions arising now in a natural way are the following:

1. Is $\|\cdot\|_2$ a norm?
2. Is $\mathbb{R}^N$, when equipped with this norm $\|\cdot\|_2$, still a Banach space?

In order to give answers to these questions, we need a result known as *Schwarz inequality*

Schwarz inequality

Proposition 219.

Hyp $x = (\xi_1, \ldots, \xi_N)$ *and* $y = (\eta_1, \ldots, \eta_N) \in \mathbb{R}^N$ *for* $N = 1, 2, 3, \ldots$.
Concl *We have*

$$\boxed{\left(\sum_{j=1}^{N} \xi_j \cdot \eta_j\right)^2 \leq \sum_{j=1}^{N} \xi_j^2 \cdot \sum_{j=1}^{N} \eta_j^2.}$$

Proof. From

$$0 \leq (a \pm b)^2 = a^2 \pm 2ab + b^2$$

we get the estimate

$$\pm 2ab \leq a^2 + b^2 \qquad \text{(for all } a \text{ and } b \in \mathbb{R}.)$$

Let us put

$$a := \frac{\xi_k}{\left(\sum_{j=1}^N \xi_j^2\right)^{1/2}} \quad \text{and} \quad b := \frac{\eta_k}{\left(\sum_{j=1}^N \eta_j^2\right)^{1/2}}$$

so that

$$\frac{\pm 2\xi_k \eta_k}{\left(\sum_{j=1}^N \xi_j^2\right)^{1/2} \cdot \left(\sum_{j=1}^N \eta_j^2\right)^{1/2}} \leq \frac{\xi_k^2}{\sum_{j=1}^N \xi_j^2} + \frac{\eta_k^2}{\sum_{j=1}^N \xi_j^2}$$

Summing over all k, we get

$$\frac{\pm 2\sum_{j=1}^N \xi_k \eta_k}{\left(\sum_{j=1}^N \xi_j^2\right)^{1/2} \cdot \left(\sum_{j=1}^N \eta_j^2\right)^{1/2}} \leq \frac{\sum_{j=1}^N \xi_k^2}{\sum_{j=1}^N \xi_j^2} + \frac{\sum_{j=1}^N \eta_k^2}{\sum_{j=1}^N \xi_j^2} = 2,$$

i.e.

$$\pm\frac{\sum_{j=1}^N \xi_k \eta_k}{\left(\sum_{j=1}^N \xi_j^2\right)^{1/2} \cdot \left(\sum_{j=1}^N \eta_j^2\right)^{1/2}} \leq 1$$

or

$$-1 \leq \frac{\sum_{j=1}^N \xi_k \eta_k}{\left(\sum_{j=1}^N \xi_j^2\right)^{1/2} \cdot \left(\sum_{j=1}^N \eta_j^2\right)^{1/2}} \leq 1.$$

Hence, squaring up, we get

$$\frac{\left(\sum_{j=1}^N \xi_k \eta_k\right)^2}{\sum_{j=1}^N \xi_j^2 \cdot \sum_{j=1}^N \eta_j^2}. \leq 1.$$

This gives the desired claim! □

We can now address the first of the above questions:

Proposition 220.

Hyp

- $N = 1, 2, 3 \ldots$
- $\|\cdot\|_2 : \mathbb{R}^N \to [0, +\infty[$ *given by*

$$\|x\|_2 := \sqrt{\sum_{j=1}^{N} \xi_j^2}, \qquad \textit{for } x = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N.$$

Concl $\|\cdot\|_2$ *is a norm on* $(\mathbb{R}^N, +.\cdot)$.

Proof.

- **Strict positivity:** We have $\|x\|_2 \geq 0$, $\forall x \in \mathbb{R}^N$. Moreover

$$\|x\|_2 = \sqrt{\sum_{j=1}^{N} \xi_j^2} = 0 \Longleftrightarrow \|x\|_\infty = \max_{j \leq 1 \leq N} |\xi_j| = 0 \Longleftrightarrow x = 0.$$

- **Homogeneity:** We have

$$\|\alpha \cdot x\|_2 = \sqrt{\sum_{j=1}^{N} (\alpha \cdot \xi_j)^2} = \sqrt{\alpha^2} \cdot \sqrt{\sum_{j=1}^{N} \xi_j^2} = |\alpha| \cdot \|x\|_2.$$

- **Triangular inequality:** We have, by Schwarz inequality,

$$\begin{aligned}
\|x + y\|_2^2 &= \sum_{j=1}^{N} \underbrace{(\xi_j + \eta_j)^2}_{=\xi_j^2 + 2\xi_j\eta_j + \eta_j^2} \\
&\leq \sum_{j=1}^{N} \xi_j^2 + 2\left(\sum_{j=1}^{N} \xi_j^2\right)^{1/2} \cdot \left(\sum_{j=1}^{N} \eta_j^2\right)^{1/2} + \sum_{j=1}^{N} \eta_j^2 \\
&= \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 \\
&= (\|x\|_2 + \|y\|^2)^2 \\
\|x + y\|_2 &\leq \|x\|_2 + \|y\|_2.
\end{aligned}$$

Hence we are done! □

We can now address the second of the above questions:

Proposition 221.

Hyp $N \in \{1, 2, 3, \ldots\}$.
Concl *The linear space* $(\mathbb{R}^N, +, \cdot)$ *equipped with the* Euclidean norm

$$\|x\|_2 := \left(\sum_{j=1}^{N} \xi_j^2\right)^{1/2}, \qquad \textit{for } x = (\xi_1, \ldots, \xi_N$$

is a Banach space.
Moreover, convergence in this Banach space means componentwise convergent in the following sense:

$$\lim_{n\to\infty} x_n = x \Longleftrightarrow \lim_{n\to\infty} \xi_j^{(n)} = \xi_j \in \mathbb{R}, \qquad \textit{for } k = 1, 2, \ldots, N,$$

where

$$x_n = (\xi_1^{(n)}, \ldots, \xi_N^{(n)}) \qquad \textit{and} \qquad x = (\xi_1, \ldots, \xi_N).$$

Before proceeding with the proof, let us remark that:

Remark 222. *In* $(\mathbb{R}^N, +, \cdot)$, *we have*

$$\lim_{n\to\infty} \|x_n - x\|_\infty = 0 \Longleftrightarrow \lim_{n\to\infty} \|x_n - x\|_2 = 0.$$

Hence, convergence means the same for both norms; in a similar way, 'to be Cauchy' means the same in both norms.

This is so in all finite dimensional Banach spaces!

Proof. We will use the following estimates:

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{N} \cdot \|x\|_\infty$$

If $|\xi_k| = \max_{1\le j\le N} |\xi_j|$, these inequalities follow from

$$\underbrace{|\xi_k|}_{=\|x\|_\infty} = \sqrt{\xi_k^2} \le \underbrace{\sqrt{\sum_{j=1}^{N} \xi_j^2}}_{=\|x\|_2} \le \sqrt{\sum_{j=1}^{N} \xi_k^2} = \sqrt{N \cdot \xi_k^2} = \sqrt{N} \cdot \underbrace{|\xi_k|}_{=\|x\|_\infty}.$$

Now, if the sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy in $(\mathbb{R}^N, \|\cdot\|_2)$ then the relation

$$\|x_n - x_m\|_\infty \le \|x_n - x_m\|_2,$$

implies that this sequence is Cauchy in $(\mathbb{R}^N, \|\cdot\|_\infty)$, too. Hence

$$\exists x \in \mathbb{R}^N \text{ with } \lim_{n\to\infty} \|x_n - x\|_\infty = 0.$$

Since

$$\|x_n - x_m\|_2 \leq \sqrt{N} \cdot \|x_n - x_m\|_\infty,$$

we have

$$\lim_{n\to\infty} \|x_n - x\|_2 = 0 \text{ , too.}$$

Hence the given Cauchy sequence converges with respect to $\|\cdot\|_2$, too. □

Definition 223.

Given: linear space $(X, +, \cdot)$ is equipped with two norms

$$\|\cdot\|_1 \quad \text{and} \quad \|\cdot\|_2.$$

we say: the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent iff:
there exist positive constants a and b such that

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_2 \qquad \forall x \in X.$$

Remark 224. *As soon as two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, the notions of 'convergent' and 'of being Cauchy' coincide.*

All norms on finite dimensional Banach spaces are equivalent

For *finite dimensional Banach spaces*, we have the following result:

Proposition 225.
All norms on a finite dimensional *Banach space are equivalent.*

In infinite dimensional Banach spaces, this is no longer true.
Recall that $(C[-1,1], \|\cdot\|_1)$ equipped with the norm

$$\|u\|_1 := \int_{-1}^{1} |u(x)|\, dx$$

is not complete and hence not a Banach space. We show now that, *if one introduces another norm on this linear space, one gets a Banach space*. At the same time, we will see that *completeness reduces, in this case, to a fundamental property of uniform converging sequences.*

Proposition 226.
The linear space $(C[a,b],+,\cdot)$ *(with* $-\infty < a < b < +\infty$*) can be equipped via*

$$\|u\|_\infty := \max_{a\le x\le b} |u(x)|$$

with a norm.
Convergence of a sequence $\{u_n(x)\}_{n=1}^\infty$ *with respect to this norm* $\|\cdot\|_\infty$ *means* uniform convergence*:*

$$\begin{gathered}\forall \varepsilon > 0, \quad \exists n_0 = n_0(\varepsilon) \text{ such that} \\ |u_n(x) - u(x)| \le \varepsilon, \forall x \in [a,b] \text{ as soon as } n \ge n_0\end{gathered}$$

(herein, n_0 *does not depend on* x*).*

Remark 227. *Any continuous function* u *defined on a bounded and closed interval* $[a,b]$ *achieves its maximum. Thus*

$$\max_{a\le x\le b} |u(x)| = \sup_{a\le x\le b} |u(x)|$$

exists as a real number, and therefore $\|\cdot\|_\infty$ *is well-defined!*

Remark 228. Point-wise convergence *means*

$$\begin{gathered}\forall \varepsilon > 0, \forall x \in [a,b], \quad \exists n_0 = n_0(\varepsilon, x) \text{ such that} \\ |u_n(x) - u(x)| \le \varepsilon \text{ as soon as } n \ge n_0\end{gathered}$$

Clearly

uniform convergence $\Longrightarrow$ *point-wise convergence*

(uniform convergence is more difficult to be achieved).

Proof.

- **Strict positivity:** We have $\|x\|_\infty \ge 0, \forall x \in C[a,b]$. Moreover

$$\|x\|_\infty = \max_{a\le x\le b} |u(x)| = 0 \Longleftrightarrow \forall x \in [a,b], \quad u(x) = 0 \Longleftrightarrow u = 0.$$

- **Homogeneity:** We have

$$\|\alpha \cdot u(x)\|_2 = \max_{a\le x\le b} |\alpha \cdot u(x)| = \max_{a\le x\le b} |\alpha| \cdot |u(x)| = |\alpha| \cdot \max_{a\le x\le b} |u(x)| = |\alpha| \cdot \|u(x)\|_\infty.$$

- **Triangular inequality:** We have

$$\forall x \in [a,b], \quad |u(x) + v(x)| \le |u(x)| + |v(x)| \le \underbrace{\max_{a\le y\le b} |u(y)|}_{=\|u(x)\|_\infty} + \underbrace{\max_{a\le y\le b} |v(y)|}_{=\|v(x)\|},$$

so that

$$\|u(x) + v(x)\|_\infty = \max_{a\le x\le b} |u(x) + v(x)| \le \|u(x)\|_\infty + \|v(x)\|_\infty$$

Hence we are done! □

The Banach space $(C[a,b], \|\cdot\|_\infty)$

Proposition 229.

Hyp

- $-\infty < a < b < +\infty$ *kept fixed*
- *the space*

$$C[a,b] := \{u : [a,b] \to \mathbb{R} \ : \ u \textit{ is continuous}\}$$

equipped with the norm $\|u\|_\infty := \max_{a\le x\le b} |u(x)|$

Concl *The space* $(C[a,b], \|\cdot\|_\infty)$ *is a Banach space.*

Proof. Let $\{u_n\}_{n=1}^{+\infty}$ be a Cauchy sequence in $(C[a,b], \|\cdot\|_\infty)$. Thus, for any given tolerance $\varepsilon > 0$, there exists a threshold $n_0 = n_0(\varepsilon)$ such that

$$\|u_n - u_m\|_\infty = \max_{a\le x\le b} |u_n(x) - u_m(x)| < \varepsilon \quad \text{as soon as } n, m \ge n_0.$$

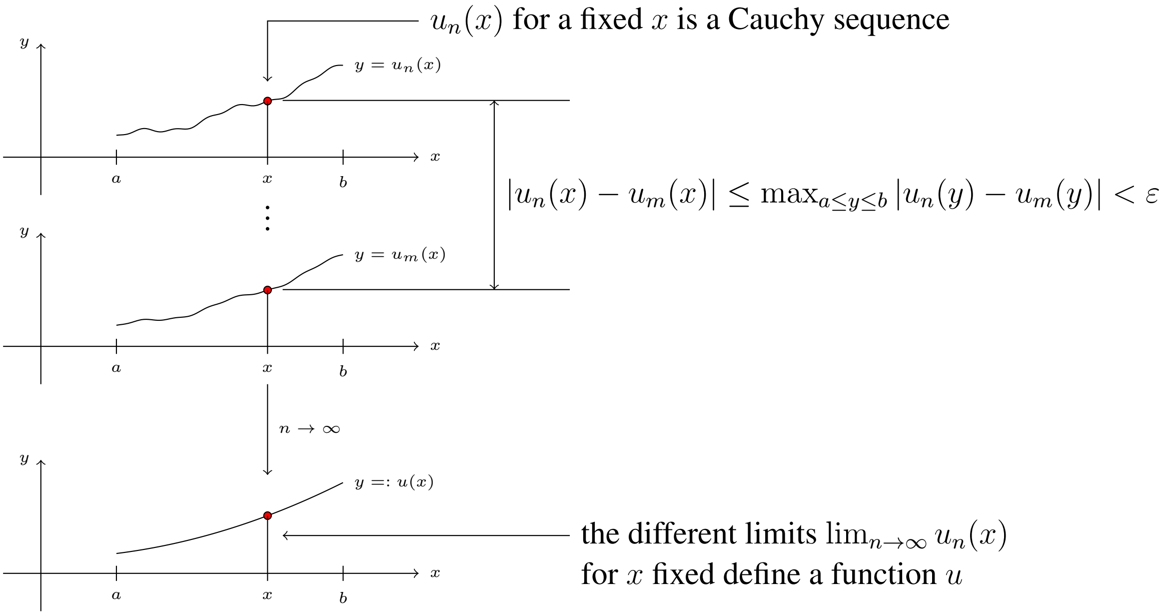
Thus, $\forall x \in [a,b]$ kept fixed,

$$|u_n(x) - u_m(x)| < \varepsilon \quad \text{as soon as } n, m \ge n_0.$$

But this means that, $\forall x \in [a,b]$ kept fixed, the sequence $\{u_n(x)\}_{n=1}^{\infty}$, as a sequence of real numbers, is a Cauchy sequence. Thus, the following limit exists for each $x \in [a,b]$:

$$\lim_{n\to\infty} u_n(x) =: u(x).$$

Remark that the value of this limit depends on x.



Recall that, $\forall x \in [a,b]$ kept fixed,

$$|u_n(x) - u_m(x)| < \varepsilon \quad \text{as soon as } n, m \geq n_0.$$

Thus, $\forall x \in [a,b]$ kept fixed,

$$|u_n(x) - u(x)| \leq \varepsilon \quad \text{as soon as } n \geq n_0,$$

i.e.

$$\|u_n - u\|_\infty = \sup_{a \leq x \leq b} |u_n(x) - u(x)| \leq \varepsilon \quad \text{as soon as } n \geq n_0.$$

This means, that the sequence of continuous functions $\{u_n(x)\}_{n=1}^{\infty}$ converges uniformly to a function $u(x)$; thus u is continuous, too.

We have established the existence of a limit $u \in C[a,b]$ for any Cauchy sequence in $C[a,b], \|\cdot\|_\infty$. Thus we are done! □

The coupling of geometry and analysis

We have used the fact that the uniform limit of a sequence of continuous functions is a continuous function.

This is a 'standard result' in analysis; for completeness, we give a proof below.

We invite the reader once more to be fascinated by the fact that *the completeness in the present case, a somewhat geometric or topological property, can be interpreted as a deep result in analysis.*

The above announced proof

Proof. Consider a sequence $\{u_n(x)\}_{n=1}^{\infty}$ of continuous functions, defined over a bounded and closed interval $[a,b]$. Suppose that this sequence converges in a uniform way; this means that, for any given tolerance $\varepsilon > 0$, there exists a threshold $n_0 = n_0(\varepsilon)$ with

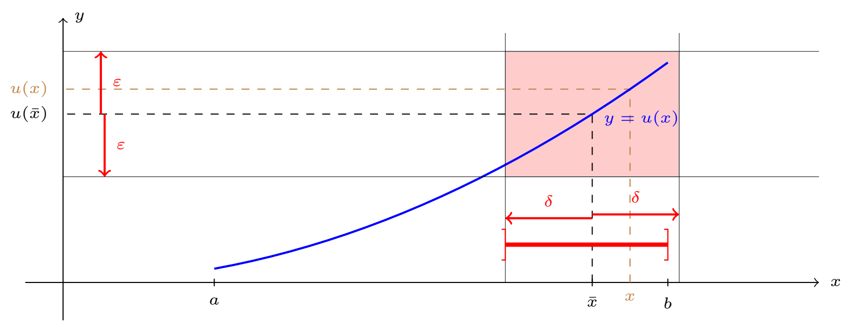
$$\max_{a \leq x \leq b} |u_n(x) - u(x)| < \varepsilon \quad \text{as soon as } n \geq n_0.$$

We show that

$$\boxed{\text{the limit function } u \text{ is continuous, too.}}$$

Consider a fixed point $\bar{x} \in [a,b]$ and let us show that the limit function u is continuous at this point $\bar{x}$. Thus, we must show that, given any tolerance $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$|u(x) - u(\bar{x})| < \epsilon, \quad \forall x \in]\bar{x} - \delta, \bar{x} + \delta[\cap [a,b].$$



So let us fix some tolerance $\varepsilon > 0$. Due to the uniform convergence, there exists a threshold $n_0 = n_0(\varepsilon/3)$ such that

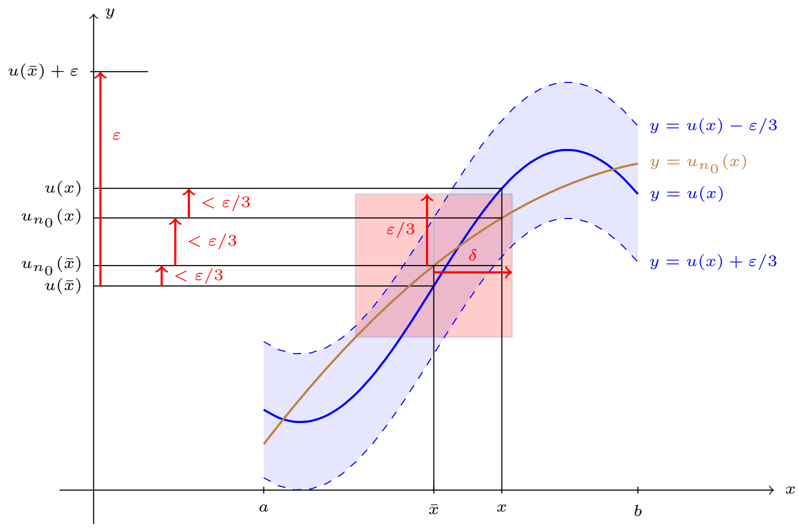
$$\max_{a \le x \le b} |u_n(x) - u(x)| < \frac{\varepsilon}{3} \qquad \text{as soon as } n \ge n_0.$$

But the function u_{n_0} is continuous; thus there exists a $\delta = \delta(\varepsilon/3)$ such that

$$|u_{n_0}(x) - u_{n_0}(\bar{x})| < \frac{\varepsilon}{3} \qquad \forall x \in]\bar{x} - \delta, \bar{x} + \delta[\cap [a, b]$$

Thus we get, $\forall x \in]\bar{x} - \delta, \bar{x} + \delta[\cap [a, b]$,

$$\begin{aligned} |u(x) - u(\bar{x})| &= |(u(x) - u_{n_0}(x)) + (u_{n_0}(x) - u_{n_0}(\bar{x})) + \\ &\quad +(u_{n_0}(\bar{x}) - u(\bar{x}))\\ &\le |u(x) - u_{n_0}(x)| + |u_{n_0}(x) - u_{n_0}(\bar{x})| + \\ &\quad +|u_{n_0}(\bar{x}) - u(\bar{x})|\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$



□

6.3. Properties of Cauchy sequences

Cauchy sequences play a major role in Banach spaces, since the concepts of

- convergent sequences and
- Cauchy sequences

coincide in such spaces.

Thus, tools allowing to check the "Cauchy property" are welcome!

Let us give such a tool.

Checking the Cauchy property

Proposition 230.

Hyp *a sequence* $\{u_n\}_{n=1}^{+\infty}$ *in a normed space* $(X, \|\cdot\|)$

Concl

1. *If*

$$\sum_{j=1}^{\infty} \|u_{j+1} - u_j\| < +\infty \quad \textit{(i.e. this series is convergent)}$$

then $\{u_n\}_{n=1}^{+\infty}$ *is a Cauchy sequence.*

2. *Hence, if* X *is a Banach space, any sequence* $\{u_n\}_{n=1}^{+\infty}$ *with*

$$\sum_{j=1}^{\infty} \|u_{j+1} - u_j\| < +\infty$$

converges.

Proof. The proof relies on the fact that, for $m > n$,

$$\begin{aligned}
\|u_n - u_m\| &\leq \|u_n - u_{n+1}\| + \|u_{n+1} - u_{n+2}\| + \\
&\quad + \|u_{n+2} - u_{n+3}\| + \cdots + \|u_{m-1} - u_m\| \\
&\leq \|u_{n+1} - u_n\| + \|u_{n+2} - u_{n+1}\| + \ldots \\
&= \sum_{j=n}^{\infty} \|u_{j+1} - u_j\| \\
&< \varepsilon
\end{aligned}$$

if n is large enough. □

When analyzing the convergence of a given Cauchy sequence $\{u_n\}_{n=1}^{+\infty}$ in a normed space (where we ignore wether or not this space is Banach, or where we know that this space is not Banach), we need the following tool in order to show that this Cauchy sequence $\{u_n\}_{n=1}^{+\infty}$ is converging.

Checking the convergence of a Cauchy sequence

Proposition 231.

Hyp $\{u_n\}_{n=1}^{+\infty}$ *is a Cauchy sequence in a normed space* $(X, \|\cdot\|)$.
Concl *If this sequence has a convergent sub-sequence* $\{u_{n_k}\}_{k=1}^{\infty}$ *with*

$$\lim_{k\to\infty} u_{n_k} = u,$$

then the whole sequence is convergent with

$$\lim_{n\to\infty} u_n = u.$$

6.4. The Banach spaces $L^p(X, \mathscr{A}, \mu)$

The L^p-spaces are Banach spaces

Proposition 232.

Hyp $1 \le p \le +\infty$
Concl *For the space*

$$L^p(X, \mathscr{A}, \mu) \qquad \textit{resp.} \qquad L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$$

are Banach spaces.
Hence, every Cauchy sequence $\{f_n\}_{n=1}^{+\infty}$ *in these spaces is convergent, i.e.,*

$$\exists f \in L^p \text{ with } \lim_{n\to\infty} f_n = f,$$

where L^p *stands for* $L^p(X, \mathscr{A}, \mu)$ *resp.* $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$.
Thereby $\lim_{n\to\infty} f_n = f$ *means*

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

We prefer the notation $f_n \to f$ *in* L^p.

Remark 233. *The proof of the above proposition gives an additional result:*
There exists a sub-sequence $\{f_{k_n}\}_{n=1}^{\infty}$ *with*

$$\lim_{n\to\infty} f_{k_n}(x) = f(x) \quad \mu\text{-a.e.}$$

Proof (For $1 \le p < \infty$*).* It is enough to show that any given Cauchy sequence $\{f_n\}_{n=1}^{+\infty}$ has a convergent sub-sequence.

The proof of this fact will be subdivided into several steps:

1. First of all, we will construct a specific sub-sequence $\{f_{n_k}\}_{k=1}^{\infty}$ with the aim to show that this sub-sequence converges in L^p.
2. In order of have an idea of the corresponding limit function f, we show that our well-chosen sub-sequence $\{f_{n_k}\}_{n=1}^{\infty}$ converges μ-a.e. to some function f.
3. Then we show that $\lim_{k\to\infty} \|f_{n_k} - f\|_p = 0$.
4. In a last step, we show that f belongs to the space L^p.

(I) The choice of the sub-sequence If $\{f_n\}_{n=1}^{+\infty}$ is a given Cauchy sequence, we may extract a sub-sequence in the following way.

Remark that, for $k = 1, 2, 3, \ldots$

$$\exists n_k := n_0\left(\frac{1}{2^k}\right) \quad \text{such that} \quad \|f_n - f_{n_k}\|_p \leq \frac{1}{2^k} \quad \text{for all } n \geq n_k.$$

We consider now in what follows the subsequence

$$\{f_{n_k}\}_{k=1}^{\infty}$$

and we show that this sub-sequence converges to some f in L^p.

(II) The μ-a.e. convergence of this sub-sequence

We claim that the sub-sequence $\{f_{n_k}\}_{k=1}^{\infty}$ converges μ-a.e. to some $f \in \overline{\mathscr{L}}(X, \mathscr{A})$.

In order to prove this, we put

$$g_k := f_{n_{k+1}} - f_{n_k}$$

i.e.

$$\begin{aligned} g_1 &= f_{n_2} - f_{n_1} \\ g_2 &= f_{n_3} - f_{n_2} \\ g_3 &= f_{n_4} - f_{n_3} \\ &\vdots \end{aligned}$$

and we remark that

$$\begin{aligned} g_1 + g_2 + g_3 &= f_{n_4} - f_{n_1} \\ \sum_{j=1}^{k} g_j &= f_{n_{k+1}} - f_{n_1} \end{aligned}$$

for $k = 1, 2, 3, \ldots$. Moreover

$$\|g_j\|_p \leq \frac{1}{2^j}$$

so that

$$\|\sum_{j=1}^{k} |g_j|\ \|_p \leq \sum_{j=1}^{k} \|\ |g_j|\ \|_p \leq \sum_{j=1}^{k} \frac{1}{2^j} < 1.$$

Hence we get, by monotone convergence

$$\begin{aligned}\| \sum_{j=1}^{\infty} |g_j| \, \|_p &= \| \lim_{k\to\infty} \sum_{j=1}^{k} |g_j| \, \|_p \\ &= \lim_{k\to\infty} \| \sum_{j=1}^{k} |g_j| \, \|_p \leq 1\end{aligned}$$

so that the series

$$\left\{ \sum_{j=1}^{k} g_j \right\}_{k=1}^{\infty} = \{ f_{n_{k+1}} - f_{n_1} \}_{k=1}^{\infty}$$

converges absolutely μ-a.e..

This imppies that the sub-sequence $\{f_{n_k}\}_{k=1}^{\infty}$ converges μ-a.e. to

$$f(x) := \left(\sum_{j=1}^{\infty} g_j(x) \right) + f_{n_1}(x).$$

(III): Moreover, $\lim_{k\to\infty} \|f_{n_k} - f\|_p = 0$**, where** f **is the abvoe defined function:**
Indeed

$$\begin{aligned}\int_X |f_{n_k}(x) - f(x)|^p \; d\mu(x) &= \int_X \left| f_{n_k} - \lim_{j\to\infty} f_{n_j} \right|^p \; d\mu \\ &= \int_X \lim_{j\to\infty} \left| f_{n_k} - f_{n_j} \right|^p \; d\mu \\ &= \int_X \liminf_{j\to\infty} \left| f_{n_k} - f_{n_j} \right|^p \; d\mu \\ &\leq \liminf_{j\to\infty} \int_X \left| f_{n_k} - f_{n_j} \right|^p \; d\mu \\ &\leq \left(\frac{1}{2^k} \right)^p \\ \lim_{k\to\infty} \left(\int_X |f_{n_k}(x) - f(x)|^p \; d\mu(x) \right)^{1/p} &\leq \lim_{k\to\infty} \frac{1}{2^k} = 0\end{aligned}$$

(IV) It remains to show that $f \in L^p(X, \mathscr{A}, \mu)$ **resp.** $f \in L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$**:**
This is indeed the case since

$$\begin{aligned}\int_X |f(x)|^p \; d\mu(x) &\leq \underbrace{\int_X |f(x) - f_{n_k}|^p \; d\mu(x)}_{\to 0 \text{ as } k\to\infty} + \int_X |f_{n_k}(x)|^p \; d\mu(x) \\ &< +\infty.\end{aligned}$$

What we have shown now is that any given Cauchy sequence $\{f_n\}_{n=1}^{+\infty}$ has a convergent sub-sequence convergint in L^p to a limit function f belonging to $L^p(X, \mathscr{A}, \mu)$ resp. $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$. Thus the whole Cauchy sequence converges to this limit funtion f in $L^p(X, \mathscr{A}, \mu)$ resp. $L^p_{\mathbb{C}}(X, \mathscr{A}, \mu)$.

This gives the desired claim! □

L^p-spaces over spaces of finite measure

Proposition 234.

<u>Hyp</u> *$(X, \mathscr{A}, \mu)$ is a measure space with $\mu(X) < +\infty$. Tyical examples are*

- *$X = [a, b]$ and $\mu = \lambda^1$ with $-\infty < a < b < +\infty$;*
- *μ a probability.*

<u>Concl</u> *Then, for $1 < p \leq \infty$, we have*

$$L^p(X, \mathscr{A}, \mu) \subset L^1(X, \mathscr{A}, \mu) \qquad \textit{resp.} \qquad L^p_{\mathbb{C}}(X, \mathscr{A}, \mu) \subset L^1_{\mathbb{C}}(X, \mathscr{A}, \mu)$$

with

$$\|f\|_1 \leq (\mu(X))^{1/q} \cdot \|f\|_p, \qquad \textit{where as usual } \frac{1}{p} + \frac{1}{q} = 1$$

(with $q = 1$ if $p = \infty$).

Proof. For $p < \infty$, the claim follows from

$$\begin{aligned} \int_X |f(x)| \; d\mu(x) &= \int_X \underbrace{1}_{\in L^q} \cdot \underbrace{|f(x)|}_{\in L^p} \; d\mu(x) \\ &\leq (\mu(X))^{1/q} \cdot \|f\|_p. \end{aligned}$$

For $p = \infty$, the claim follows from $|f| \leq \|f\|_\infty$ μ-a.e. and

$$\begin{aligned} \|f\|_1 &= \int_X |f| \; d\mu(x) \\ &\leq \int_X \|f\|_\infty \; d\mu(x) = \|f\|_\infty \int_X \; d\mu(x) \\ &= \|f\|_\infty \cdot \mu(X). \end{aligned}$$

□

7

Operators and fixed points

7.1. Operators as mappings

Definition 235.
An operator

$$A : M \subset X \to Y, u \mapsto Au$$

is a mapping with domain M, target Y and range $A(M) \subset Y$.
Remark that $M = X$ is possible.
We introduce the notations

$$\mathrm{dom}(A) := M$$

for the domain and

$$\mathrm{Ran}(A) := A(M)$$

for the range of an operator A.

Definition 236.
A functional f is an operator with target $\mathbb{K}$:

$$f : M \subset X \to \mathbb{K}.$$

A first kind of integral operators

Example 237.
Suppose that $-\infty < a < b < +\infty$.
We may consider the operator

$$A : C[a, b] \to C[a, b], \quad u \mapsto Au$$

defined by

$$(Au)(x) := \int_a^b F(x, y, u(y)) \ dy, \qquad (x \in [a, b]),$$

where

$$F : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$$

is a given continuous function.
Remark that inside the above integral, x plays the role of a parameter: the resulting integral thus depends on x.
Remark that, due to the continuity of the so called kernel F, the resulting function

$$x \mapsto (Au)(x) = \int_a^b F(x, y, u(y)) \ dy, \qquad (x \in [a, b]),$$

is continuous if the function

$$u : [a, b] \to \mathbb{R}$$

is continuous.

Such operators are called integral operators.

A second kind of integral operators

Example 238.

Suppose that $-\infty < a < b < +\infty$.

We may consider the operator

$$B : C[a, b] \to C[a, b], \quad u \mapsto Bu$$

defined by

$$(Bu)(x) := \int_a^x F(x, y, u(y)) \, dy, \qquad (x \in [a, b]),$$

where

$$F : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$$

is a given continuous function.

Remark that inside the above integral, x plays the role of a parameter: the resulting integral thus depends on x.

Remark that, due to the continuity of the so called kernel F, the resulting function

$$x \mapsto (Bu)(x) = \int_a^x F(x, y, u(y)) \, dy, \qquad (x \in [a, b]),$$

is continuous if the function

$$u : [a, b] \to \mathbb{R}$$

is continuous.

Remark that this kind of integral operators can be considered as a special case of the integral operators considered in the previous example.

7.2. Banach's fixed point theorem

7.2.1. Fixed points

Fixed points problems may appear in a natural way

Let us consider an operator $B : M \subset X \to Y$ and the generic problem of "solving the equation"

$$Bu = v,$$

where $v \in Y$ is given and where we are looking for (at least) one solution $u \in M$.

If $X = Y$, this equation may be written as

$$(Bu - v) + u = u.$$

So, if one sets

$$A : M \subset X \to X, \quad u \mapsto Au := Bu - v + u,$$

the above equation reduces to a fixed point problem

$$Au = u \qquad \text{for } u \in M.$$

7.2.2. Fixed points obtained by iteration

Solutions of a fixed point problem may be obtained by iteration

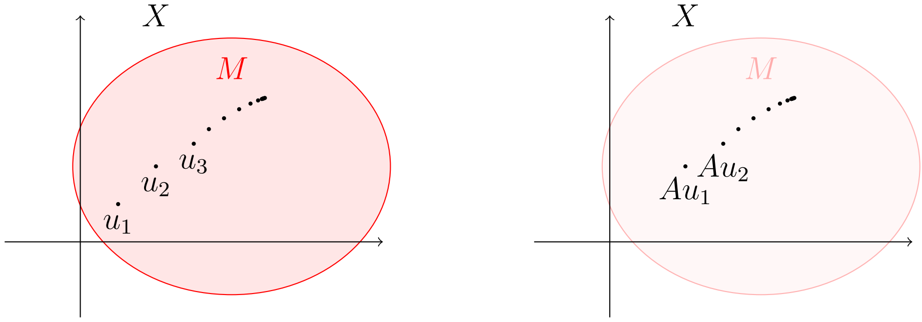
So let us consider a fixed point problem

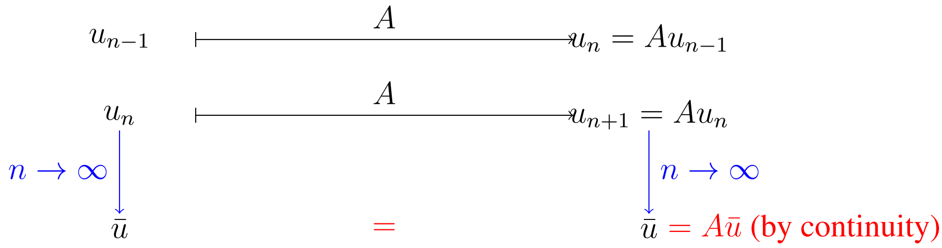
$$Au = u,$$

where $A : M \subset X \to X$ is a given operator. We consider the following process:

$$\begin{cases} \text{choose any starting point } u_0 \in M \\ \text{compute in an iterative way} \quad u_{n+1} := Au_n \quad , \text{for } n = 1, 2, 3, \ldots. \end{cases}$$

We expect, under suitable conditions, that the above process gives us a sequence $\{u_n\}_{n=1}^{+\infty}$ converging to a fixed point of A.





A first necessary condition

In order to be well-defined, the above process

$$\begin{cases} u_0 \in M \\ u_{n+1} := Au_n \quad \text{, for } n = 1, 2, 3, \ldots . \end{cases}$$

can be used only if

$$A : M \subset X \to M.$$

Remark 239. *Since we expect that the above defined sequence $\{u_n\}_{n=1}^{+\infty}$ converges to a fixed point $\bar{u}$, it may be wise to impose the condition that M is closed.*

Remark 240. *We will be able to show that the above defined sequence $\{u_n\}_{n=1}^{+\infty}$ is, under some conditions, a Cauchy sequence, and thus*

$$\exists \bar{u} := \lim_{n\to\infty} u_n \in M$$

if we assume that X is a Banach space.

Remark 241. *As soon as the convergence $\lim_{n\to\infty} u_n = \bar{u} \in M$ is established, we can take the limit in*

$$u_{n+1} = Au_n$$

and we get

$$\bar{u} = \lim_{n\to\infty} Au_n = A \lim_{n\to\infty} u_n = A\bar{u},$$

i.e. $\bar{u}$ is a fixed point; however, the above arguments is only valid if the operator A is continuous.

Collecting all these remarks, we are ready for Banach's fixed point theorem.
Remark that we *still have no idea* how to guarantees that the above sequence $\{u_n\}_{n=1}^{+\infty}$ with

$$\begin{cases} u_0 \in M \\ u_{n+1} := Au_n \quad \text{, for } n = 1, 2, 3, \ldots . \end{cases}$$

is a Cauchy sequence.

7.2.3. Fixed points of contractive operators

Proposition 242.
[Banach's fixed point theorem (1920)]

<u>*Hyp*</u> *Suppose that* $(X, \|\cdot\|)$ *is a Banach space and that*

$$M \subset X \qquad \textit{(with } M \neq \varnothing\textit{)}$$

is closed.
Consider an operator

$$A : M \to M, \quad u \in M \mapsto Au \in M$$

that is k*-contractive with* $0 \leq k < 1$*; by this we mean that*

$$\|Au - Av\| \leq k \cdot \|u - v\|, \qquad \forall u, v \in M.$$

<u>*Concl*</u>

1. ***Existence and uniqueness:***
   There exists exactly one fixed point $\bar{u} \in M$*:*

   $$\exists! \bar{u} \in M \quad \textit{with} \quad A\bar{u} = \bar{u}.$$

2. ***Convergence of the iteration process:***
   $\forall u_0 \in M$*, the sequence* $\{u_n\}_{n=1}^{+\infty}$ *defined by*

   $$u_{n+1} := Au_n \qquad \textit{, for } n = 1, 2, 3, \ldots$$

   converges to the unique fixed point $\bar{u}$*. Moreover*

   $$\|u_n - \bar{u}\| \leq \frac{k^n}{1-k} \|u_1 - u_0\|, \qquad \textit{for } n = 1, 2, 3, \ldots$$

   (a-priori estimate for the speed of convergence).

Proof. **(I) Uniqueness of the fixed point:**
Suppose that

$$Au = u \qquad \text{and} \qquad Av = v.$$

Then

$$\|u - v\| = \|Au - Av\| \leq k \cdot \|u - v\|$$

i.e.

$$\underbrace{(1-k)}_{>0} \cdot \|u - v\| \leq 0$$

so that

$$\|u - v\| = 0.$$

Hence we get $u = v$, and this shows that the fixed point is unique.

(II) The recursively constructed sequence $\{u_n\}_{n=1}^{+\infty}$ is a Cauchy sequence:

For $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|Au_n - Au_{n-1}\| \leq k \cdot \|u_n - u_{n-1}\| \\ &= k \cdot \|Au_{n-1} - Au_{n-2}\| \leq k^2 \cdot \|u_{n-1} - u_{n-2}\| \\ &\vdots \\ &\leq k^n \cdot \|u_1 - u_0\| \end{aligned}$$

Hence, for $m = 1, 2, 3, \ldots$,

$$\begin{aligned} \|u_{n+m} - u_n\| &= \|(u_{n+m} - u_{n+m-1}) + (u_{n+m-1} - u_{n+m-2}) + \cdots \\ &\qquad \cdots + (u_{n+1} - u_n)\| \\ &= \|\sum_{j=}^{m} (u_{n+j} - u_{n+j-1})\| \\ &\leq \sum_{j=}^{m} \|u_{n+j} - u_{n+j-1}\| \leq \sum_{j=1}^{m} k^{n+j-1} \|u_1 - u_0\| \\ &\leq k^n \left(\sum_{j=0}^{\infty} k^j \right) \|u_1 - u_0\| = k^n \cdot \frac{1}{1-k} \cdot \|u_1 - u_0\|. \end{aligned}$$

This shows that the sequence $\{u_n\}_{n=1}^{+\infty}$ is a Cauchy sequence.

Hence this sequence converges (since X is a Banach space) and the limit point

$$\bar{u} := \lim_{n \to \infty} u_n \in M$$

(since M is closed).

(III) Existence of a fixed point: $\bar{u}$ is a fixed point.

Indeed, the estimate

$$\|Au_n - A\bar{u}\| \leq k \cdot \|u_n - \bar{u}\|$$

shows that

$$\lim_{n \to \infty} Au_n = A\bar{u}.$$

Remark that this simply means that the operator A is continuous.

Taking the limit in

$$u_{n+1} = Au_n$$

we get

$$\bar{u} = A\bar{u},$$

i.e. $\bar{u}$ is a fixed point of A.

(IV) The a-priory estimate:

Taking the limit $m \to \infty$ in the above derived estimate

$$\|u_{n+m} - u_n\| \leq \frac{k^n}{1-k} \cdot \|u_1 - u_0\|$$

we get the desired estimate

$$\|\bar{u} - u_n\| \leq \frac{k^n}{1-k} \cdot \|u_1 - u_0\|$$

□

7.2.4. An application to ODE

The initial value problem

We consider the initial value problem

$$\begin{cases} \dot{u}(t) = f(t, u(t)) & \text{, for } t \in [0, b] \\ u(0) = u_0 \end{cases}$$

(with $b > 0$ fixed), where the *initial condition* $u_0 \in \mathbb{R}$ as well as the *continuous function*

$$f : [0, b] \times \mathbb{R} \to \mathbb{R}, (t, u) \mapsto f(t, u)$$

are given.

We are looking for a function

$$u : [0, b] \to \mathbb{R}, t \mapsto u(t)$$

such that

$$\dot{u}(t) = f(t, u(t)), \qquad \forall t \in [0, b].$$

An equivalent formulation of the initial value problem

An equivalent formulation of this initial value problem is

$$\boxed{\begin{array}{l} \text{Find } u \in C[0, b] \text{ such that} \\ u(t) = u_0 + \displaystyle\int_0^t f(\tau, u(\tau))\, d\tau, \quad \forall t \in [0, b]. \end{array}}$$

Therefore, we introduce the operator

$$\begin{array}{l} A : C[0, b] \to C[0, b], \quad u \mapsto Au \\ (Au)(t) := u_0 + \int_0^t f(\tau, u(\tau))\, d\tau, \quad \forall t \in [0, b]. \end{array}$$

As usual, we equip the space $C[0, b]$ with the norm

$$\|u\|_\infty := \max_{t \in [0, b]} |u(t)|$$

and we recall that $(C[0, b], \|\cdot\|_\infty)$ is a Banach space.

Final formulation of the initial value problem

Given:

- a *continuous* function $f : [0, b] \times \mathbb{R} \to \mathbb{R}$
- an *initial condition* $u_0 \in \mathbb{R}$.

Find: a fixed point

$$A\bar{u} = \bar{u}, \qquad \bar{u} \in C[0, b]$$

of the mapping

$$\begin{cases} A : C[0, b] \to C[0, b], \quad u \mapsto Au \\ (Au)(t) := u_0 + \int_0^t f(\tau, u(\tau))\, d\tau, \quad \forall t \in [0, b]. \end{cases}$$

We can apply Banach's fixed point theorem if the operator A is k-contractive with $0 \leq k < 1$.

Remark that

$$\begin{aligned} \|Au_1 - Au_2\|_\infty &= \sup_{0 \leq t \leq b} \left| \int_0^t f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))\, d\tau \right| \\ &\leq \sup_{0 \leq t \leq b} \int_0^t \underbrace{|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))|}_{\geq 0}\, d\tau \\ &\leq \int_0^b |f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))|\, d\tau \end{aligned}$$

In order to proceed, we need an additional hypothesis on f!

An additional hypothesis of f

Let us assume that the given function

$$f : [0, b] \times \mathbb{R} \to \mathbb{R}, (\tau, u) \mapsto f(\tau, u)$$

is

1. continuous in the first variable τ and
2. *Lipschitz continuous in the second variable*:

$$\begin{aligned} &\exists L > 0 \text{ with} \\ &\quad |f(\tau, u_1) - f(\tau, u_2)| \leq L \cdot |u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbb{R}, \forall \tau \in [0, b] \end{aligned}$$

Let us remark that the above Lipschitz-condition holds if for example f has a continuous derivative with respect to the second variable u with

$$\left| \frac{\partial}{\partial u} f(\tau, u) \right| \text{ is bounded for } (\tau, u) \in [0, b] \times \mathbb{R}.$$

We can now proceed in the above computations:

$$\begin{aligned}\|Au_1 - Au_2\|_\infty &\leq \int_0^b \underbrace{|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))|}_{\leq L\cdot|u_1(\tau)-u_2(\tau)|} \, d\tau \\ &\leq L \cdot \int_0^b \underbrace{|u_1(\tau) - u_2(\tau)|}_{\leq \max_{0\leq x\leq b}|u_1(x)-u_2(x)|} \, d\tau \\ &\leq L \cdot \|u_1 - u_2\|_\infty \cdot \int_0^b d\tau \\ &= L \cdot b \cdot \|u_1 - u_2\|_\infty .\end{aligned}$$

If $L \cdot b < 1$ i.e. if $b > 0$ is small enough, we can apply Banach's fixed point theorem and we get a unique solution to our initial value problem.

We formulate this result in a proposition

Theorem of Picard-Lindelöf

Proposition 243.

Hyp *Suppose given:*

- *a function* $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ *with*
  1. *continuous in the first variable* τ *and*
  2. *Lipschitz continuous in the second variable:*

     $\exists L > 0$ *with*
     $$|f(\tau, u_1) - f(\tau, u_2)| \leq L \cdot |u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbb{R}, \forall \tau \in [0, b]$$
- *an* initial condition $u_0 \in \mathbb{R}$.

Concl *The initial value problem*

$$\begin{cases} \dot{u}(t) = f(t, u(t)) & \text{, for } t \in [0, b] \\ u(0) = u_0 \end{cases}$$

has exactly one solution provided that $b \cdot L < 1$. *This last condition can be satisfied by reducing, if necessary,* $b > 0$.

7.3. Continuous operators

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces over $\mathbb{K}$, and let

$$A : M \subset X \to Y, \quad u \mapsto Au$$

be an operator.

The notion of continuity

Definition 244.

1. The operator $A : M \subset X \to Y$ is *sequentially continuous* if :

   for each converging sequence $\{u_n\}_{n=1}^{+\infty}$ in M with:

   $$\lim_{n\to\infty} u_n = u, \quad u \in M$$

   we have

   $$Au = A(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} Au_n.$$

2. The operator $A : M \subset X \to Y$ is *continuous* if :

   $$\begin{array}{l} \forall u \in M, \quad \forall \varepsilon > 0 \\ \exists \delta := \delta(u,\varepsilon) > 0 \text{ such that} \\ \quad \left. \begin{array}{l} \|u - v\|_X < \delta(u,\varepsilon) \\ v \in M \end{array} \right\} \Longrightarrow \|Au - Av\|_Y < \varepsilon. \end{array}$$

   If the threshold $\delta(u, \varepsilon)$ can be chosen in such a way that

   $$\delta := \delta(\varepsilon)$$

   does not depend on u (same value for all $u \in M$!), then A is said to be *uniformly continuous*.

Remark 245. *In normed spaces, both notions of continuity coincide:*

$$A \textit{ sequentially continuous} \iff A \textit{ continuous}$$

In a more general setting of a topological space, this is no longer the case!

k contractive operators are continuous

Proposition 246.
Every operator

$$A : M \subset X \to X, \qquad (X, \|\cdot\|_X \text{ a normed space})$$

that is k-contractive in the sense that

$$\forall u, v \in M, \qquad \|Au - Av\|_X \leq k \cdot \|u - v\|_X$$

(with a fixed $k \geq 0$) is uniformly continuous.

Proof.

$$\delta(\varepsilon) = \frac{\varepsilon}{k+1}.$$

□

Continuity is preserved under composition

Proposition 247.

<u>*Hyp*</u> *Suppose that X, Y and Z are normed spaces and consider the operators*

$$A : M \subset X \to Y \qquad \text{and} \qquad B : A(M) \subset Y \to Z.$$

We can the consider the composed operator

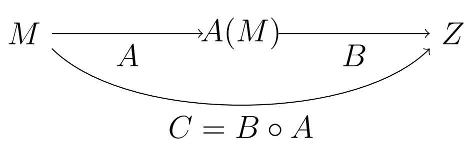
$$C := B \circ A : M \subset X \to Z$$

defined by

$$Cu := B\,(Au)\,, \qquad \forall u \in M.$$

<u>*Concl*</u>

$$\left.\begin{array}{l} A \text{ continuous} \\ B \text{ continuous} \end{array}\right\} \Longrightarrow C = B \circ A \text{ continuous}$$



Proof. Consider a sequence $\{u_n\}_{n=1}^{+\infty}$ in M with

$$\lim_{n\to\infty} u_n = u, \qquad \in M.$$

Then

- Since A is continuous, we have

$$Au = A(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} Au_n.$$

- Since B is continuous, this implies

$$B(A(u)) = B(A(\lim_{n\to\infty} u_n)) = B(\lim_{n\to\infty} Au_n) = \lim_{n\to\infty} B(A(u_n)).$$

This gives the desire claim. □

The notion of homeomorphism

Definition 248.

Given: subsets M and N of normed spaces and an operator

$$A : M \to N$$

we say: A is a *homeomorphism* iff:

1. A is a bijection; thus there exists an operator $A^{-1} : N \to M$ with
$$A^{-1}y = x \iff Ax = y.$$
2. both A and A^{-1} are continuous.

Remark 249. *The subsets M and N are said to be homeomorphic if such an operator A exists.*

7.4. Convexity

7.4.1. Convex sets

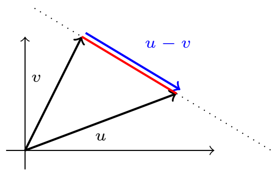
Definition 250.
Given: a set M in a linear space X (over $\mathbb{K}$)
we say: M is *convex* iff:

$$u, v \in M \Longrightarrow \alpha \cdot u + (1-\alpha)v \in M, \quad \forall \alpha \in [0,1].$$

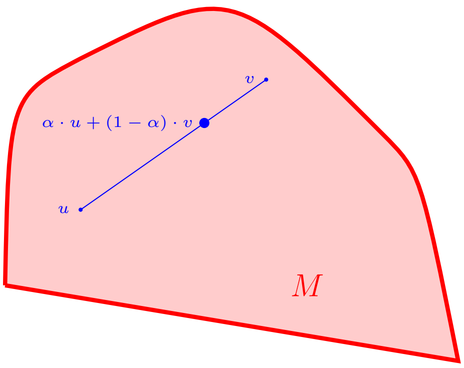
Remark 251. *Remark that*

$$\alpha \cdot u + (1-\alpha)v = v + \alpha \cdot (u - v).$$

Thus, $\alpha \cdot u + (1-\alpha)v$, with $\alpha \in [0,1]$ is the line segment between u and v.



Remark 252. *A set M is thus convex, if the segment joining any two given points u and $v \in M$ remains in M:*



Example 253.
Consider, in a normed space $(X, \|\cdot\|)$, the closed ball

$$B_r(u_0) := \{u \in X \ : \ \|u - u_0\| \leq r\}$$

with $r \geq 0$ and $u_0 \in X$ kept fixed. We show that

$$\boxed{B_r(u_0) \text{ is a convex set.}}$$

Proof. Let u, $v \in B_r(u_0)$; thus

$$\|u - u_0\| \leq r \qquad \text{and} \qquad \|v - u_0\| \leq r.$$

Then, $\forall \alpha \in [0,1]$, we have

$$\begin{aligned} \|(\alpha u + (1-\alpha)v) - u_0\| &= \|\alpha \cdot (u - u_0) + (1-\alpha)\cdot(v - u_0)\| \\ &\leq \alpha\|u - u_0\| + (1-\alpha)\|v - u_0\| \\ &\leq \alpha \cdot r + (1-\alpha)\cdot r = r \end{aligned}$$

so that

$$\alpha u + (1-\alpha)v \in B_r(u_0).$$

Thus $B_r(u_0)$ is convex. □

Remark 254. *A similar computation shows that ε-neighborhoods*

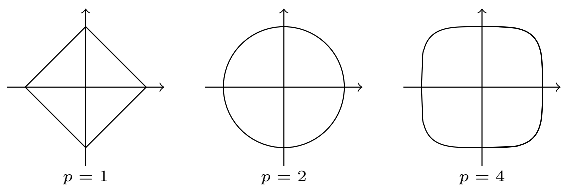
$$U_\varepsilon(u_0) := \{u \in X \ : \ \|u - u_0\| < \varepsilon\}$$

(with $\varepsilon > 0$ and $u_0 \in X$ kept fixed) are convex, too.

Example 255.
In $\mathbb{R}^N$, on may consider, for $p \in [1, +\infty[$, the norm

$$\|x\|_p := \sqrt[p]{\sum_{k=1}^{N} |\xi_k|^p}, \qquad x = (\xi_1, \xi_2, \ldots, \xi_N).$$

Depending on p, the balls $B_1(0)$ have the following shapes in $\mathbb{R}^2$:



Remark, these balls are convex.

Proposition 256.
M convex $\Longrightarrow$ $\overline{M}$ convex.

7.4.2. Convex functionals

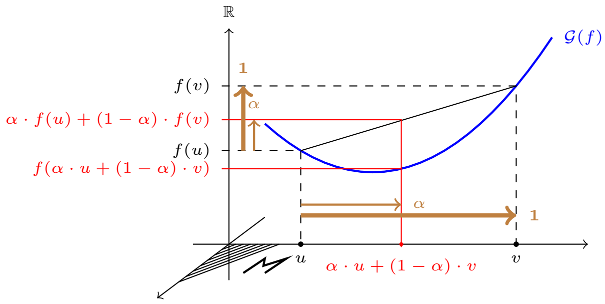
Definition 257.

Given: a functional $f : M \to \mathbb{R}, \quad x \mapsto f(x)$

we say: f is *convex* iff:

1. the set M is convex (in a linear space X over $\mathbb{K}$);
2. For all u and $v \in M$,

$$f(\alpha u + (1-\alpha)\cdot v) \leq \alpha f(u) + (1-\alpha)\cdot f(v), \quad \forall \alpha \in [0,1].$$



Example 258.

Consider a normed space $(X, \|\cdot\|)$.

Then, the norm

$$\|\cdot\| : X \to [0, +\infty[, \quad u \mapsto \|u\|$$

is continuous and convex.

Indeed, $\forall \alpha \in [0,1]$,

$$\|\alpha u + (1-\alpha)v\| \leq \alpha\|u\| + (1-\alpha)\|v\|.$$

7.5. Compactness

It is well known that on the real line $\mathbb{R}$, any bounded sequence has a converging sequence.

This property is frequently used in real analysis in one variable.

This motivates the following definitions.

7.5.1. Compact sets

Definition 259.
Let M be a subset of a normed space X.

1. M is *relatively (sequentially) compact iff*:

   every sequence $\{u_n\}_{n=1}^{+\infty}$ in M has a convergent sub-sequence $\{u_{n_k}\}_{k=1}^{+\infty}$:
   $$\lim_{k\to\infty} u_{n_k} = u.$$
   Remark that $u \in X$, but both $u \in M$ and $u \notin M$ are possible.

2. M is *(sequentially) compact* iff:

   every sequence $\{u_n\}_{n=1}^{+\infty}$ in M has a convergent sub-sequence $\{u_{n_k}\}_{k=1}^{+\infty}$:
   $$\lim_{k\to\infty} u_{n_k} = u$$
   with $u \in M$.
   Remark that, by definition, every relatively compact and closed set M is compact.

Definition 260.
Given: a subset M of a normed space X
we say: M is *bounded* iff:

$$\exists R > 0 \quad \text{such that} \quad \|u\| \leq R, \forall u \in M.$$

Compact sets in finite dimension

Proposition 261.
Any bounded and closed *set $M \subset \mathbb{K}^N$ $(N = 1, 2, 3, \ldots)$ is compact.*

Remark 262. *As we will see it below, this is no longer true in infinite-dimensional spaces.*

7.5.2. Minimizsation of functionals

Consider a *continuous* functional

$$f : M \subset X \to \mathbb{R}, \qquad u \mapsto f(u)$$

defined on a subset M of the normed space $(X, \|\cdot\|)$.

The question of the existence of a minimizer

Question: Does there exists some element $\bar{u} \in M$ such that

$$f(\bar{u}) \leq f(u), \quad \forall u \in M$$

i.e., some element $\bar{u} \in M$ with

$$f(\bar{u}) = \inf_{u \in M} f(u)?$$

Answer: Let us start with a minimizing sequence

We put

$$\alpha := \inf_{u \in M} f(u) \qquad (\in \mathbb{R} \cup \{-\infty\})$$

and we consider a so called *minimizing sequence* $\{u_n\}_{n=1}^{+\infty}$ in M. Thus we consider a sequence such that

$$f(u_n) \text{ is non-increasing with } \lim_{n \to \infty} f(u_n) = \alpha.$$

Answer: we need additional assumptions

In order to proceed, we assume that

$$\boxed{\{u_n\}_{n=1}^{+\infty} \text{ has a convergent sub-sequence.}}$$

Without loss of generality, we denote this sub-sequence by $\{u_n\}_{n=1}^{+\infty}$ again, and we put

$$\bar{u} := \lim_{n \to \infty} u_n.$$

Let us assume moreover that

$$\boxed{\bar{u} \in M.}$$

This is the case if the subset M is closed.

Answer

Thus, by the assumed continuity of f, we get

$$f(\bar{u}) = \lim_{n \to \infty} f(u_n) = \alpha = \inf_{u \in M} f(u).$$

Answer: Conclusion

Thus we may conclude that a minimizer exists under the above given assumptions.
Let us formulate this result in a proposition.

Proposition 263.

<u>*Hyp*</u> *Consider a* continuous *functional*

$$f : M \subset X \to \mathbb{R}, \qquad u \mapsto f(u)$$

defined on a subset M of the normed space $(X, \|\cdot\|)$.
Assume that M is compact, *i.e.*

- *every sequence $\{u_n\}_{n=1}^{+\infty}$ in M has a convergent sub-sequence,*
- *whose limit point $\bar{u}$ belongs to M.*

<u>*Concl*</u> *The functional f admits its infimum:*

$$\exists \bar{u} \in M \text{ with } f(\bar{u}) = \inf_{u \in M} f(u).$$

On short we say

$$\exists \bar{u} \in M \text{ with } f(\bar{u}) = \min_{M} f.$$

Corollary 264.

<u>*Hyp*</u> *Consider a* continuous *function*

$$f : M \subset X \to \mathbb{R}, \qquad u \mapsto f(u)$$

defined on a subset M of the finite-dimensional *normed space $(\mathbb{K}^N, \|\cdot\|)$ with $N = 1, 2, 3, \ldots$.*
Assume that

- *M is* bounded
- *and* closed.

<u>*Concl*</u> *The function f admits its infimum:*

$$\exists \bar{u} \in M \text{ with } f(\bar{u}) = \inf_{u \in M} f(u).$$

7.5.3. Compactness in infinite-dimensional spaces

In infinite dimensional spaces, there exist subsets M that are bounded and closed, but that are not compact.

We given now an example of such a set M in $C^1[a,b]$.

In the next subsection, we will characterize relatively compact sets in $C[a,b]$ (Theorem of Arzelà-Ascoli).

Proposition 265.

Hyp *Consider the linear space (over $\mathbb{R}$)*

$$C^1[a,b] := \{u : [a,b] \to \mathbb{R} \ : \ u \text{ has continuous derivatives on } [a,b]\}$$

(with $-\infty < a < b < +\infty$), equipped with

$$\begin{aligned} \|u\|_1 &:= \|u\|_\infty + \|u'\|_\infty \\ &= \max_{a \le x \le 1} |u(x)| + \max_{a \le x \le 1} |u'(x)|. \end{aligned}$$

Concl *$\|\cdot\|_1$ is a norm and $(C^1[a,b], \|\cdot\|_1)$ is a Banach space.*

Proof. We will only show that $(C^1[a,b], \|\cdot\|_1)$ is complete, and we leave it to the reader, to check that $\|\cdot\|_1$ is a norm.

So let $\{u_n\}_{n=1}^{+\infty}$ be a Cauchy sequence in $C^1[a,b]$, and let us show that this sequence has a limit point in $C^1[a,b]$.

To be a Cauchy sequence means

$$\begin{aligned} &\forall \varepsilon > 0 \\ &\quad \exists n_0 = n_0(\varepsilon) \text{ such that} \\ &\quad\quad \|u_n - u_m\|_1 < \varepsilon \text{ as soon as } n, m \ge n_0. \end{aligned}$$

Recalling that $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$, we may conclude that both

$$\{u_n\}_{n=1}^{+\infty} \quad \text{and} \quad \{u_n'\}_{n=1}^{+\infty}$$

are Cauchy sequences in the Banach space $(C[a,b], \|\cdot\|_\infty)$.

Thus, both sequences converge in $(C[a,b], \|\cdot\|_\infty)$:

$$\begin{aligned} &\exists u \in C[a,b] \quad \text{such that} \quad \lim_{n\to\infty} \|u_n - u\|_\infty = 0 \\ &\exists v \in C[a,b] \quad \text{such that} \quad \lim_{n\to\infty} \|u_n' - v\|_\infty = 0 \end{aligned}$$

If we can show that $u' = v$, then we have

$$\lim_{n\to\infty} \|u_n - u\|_1 = 0 \qquad \text{with } u \in C^1[a,b];$$

this is the desired result that $(C^1[a,b], \|\cdot\|_1)$ is complete.

We can show that $u' = v$ in the following way. We have

$$u_n(x) = u_n(a) + \int_a^x u_n'(\xi)\,d\xi, \qquad \forall x \in [a,b].$$

As a convergent sequence, the sequence $\{u_n'\}_{n=1}^{+\infty}$ is bounded in $(C[a,b], \|\cdot\|_\infty)$, say

$$\|u_n\|_\infty = \max_{a\le x\le b} |u_n(x)| \le R, \qquad \forall n$$

Using the majorating function $w(x) \equiv R$, we may conclude that

$$\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} u_a(a) + \int_a^x \lim_{n\to\infty} u_n'(\xi)\,d\xi, \qquad \forall x \in [a,b]$$

i.e.

$$u(x) = u(a) + \int_a^x v(\xi)\,d\xi, \qquad \forall x \in [a,b],$$

so that $u'(x) \equiv v(x)$. □

We give now an example of a subset M in $C^1[-1,1]$ that is

- bounded and
- closed,

but which is

- not compact.

Our set M will be given by

$$M := \{u \in C^1[-1,1]\ ;\ u(-1) = u(1) = 1, \|u\|_1 \le 6.\}$$

This set M is bounded by definition; moreover it is closed since

$$\lim_{n\to\infty} \|u_n - u\|_1 = 0$$

with $u_n \in M$ implies

- $u(1) = \lim_{n\to\infty} u_n(1) = 1$ and $u(-1) = \lim_{n\to\infty} u_n(-1) = 1$;
- $\|u\|_\infty = \lim_{n\to\infty} \|u_n\|_\infty \le 6$.

Bounded and closed sets in an infinite-dimensional space must not be necessarily be compact

Proposition 266.
The set

$$M := \{u \in C^1[-1,1]\ ;\ u(-1) = u(1) = 1, \|u\|_1 \leq 6.\}$$

in the Banach space $(C^1[a,b], \|\cdot\|_1)$ *is not compact, though it is bounded and closed.*

The proof will be "indirect" in the sense that we will construct a continuous functional

$$f : M \to \mathbb{R}, \quad u \mapsto f(u)$$

that has no minimizer in M. Thus M cannot be compact!

Proof. Let us consider the *continuous* functional

$$f : M \to \mathbb{R}, \quad u \mapsto f(u)$$

defined by

$$f(u) := \int_{-1}^{1} \left(1 - u'(x)^2\right)^2 \, dx.$$

(I) We show that $f(u) > 0$**,** $\forall u \in M$**:**
Since

$$\left(1 - u'(x)^2\right)^2 \geq 0, \qquad \forall x \in [-1,1],$$

we have $f(u) \geq 0$, $\forall u \in M$.
Moreover, $f(u) = 0$ would imply that

$$\left(1 - u'(x)^2\right)^2 = 0, \qquad \forall x \in [-1,1],$$

i.e. either $u'(x) \equiv 1$ or $u'(x) \equiv -1$.
But both of these conditions are incompatible with the requirement that $u(-1) = u(1) = 1$.
Thus f remains strictly positive.
(II) We show that $\inf_{u \in M} f(u) = 0$**:**
To perform this, we consider a sequence $\{u_n\}_{n=1}^{+\infty}$ in $C^1[-1,1]$ defined by

$$u_n(x) := \sqrt{\frac{n}{n+1}} \cdot \sqrt{x^2 + \frac{1}{n}}$$

Remark that

- $u_n(\pm 1) = 1$;
- $\|u_n\|_\infty \leq \sqrt{2}$;
- $u_n'(x) = \sqrt{\frac{n}{n+1}} \cdot \frac{x}{\sqrt{x^2 + \frac{1}{n}}}$, so that

$$|u_n'(x)| = \sqrt{\frac{n}{n+1}} \cdot \frac{|x|}{\sqrt{x^2 + 1/n}} \leq \sqrt{\frac{n}{n+1}} \cdot \frac{\sqrt{x^2 + 1/n}}{\sqrt{x^2 + 1/n}} < 1;$$

- $\|u_n\|_1 \leq \sqrt{2} + 1 < 6$.

Thus, $u_n \in M$ for all $n = 1, 2, 3, \ldots$. Moreover,

$$\begin{aligned} f(u_n) &= \int_{-1}^{1} \left(1 - \frac{n}{n+1} \cdot \frac{x^2}{x^2 + \frac{1}{n}}\right)^2 dx \\ &= \int_{-1}^{1} \left(\frac{(n+1)(x^2 + \frac{1}{n}) - nx^2}{(n+1)(x^2 + \frac{1}{n})}\right)^2 dx \\ &= \int_{-1}^{1} \left(\frac{x^2 + \frac{n+1}{n}}{(n+1)(x^2 + \frac{1}{n})}\right)^2 dx \\ &= \int_{-1}^{1} \underbrace{\left(\frac{\frac{x^2}{n+1} + \frac{1}{n}}{x^2 + \frac{1}{n}}\right)^2}_{\leq 1} dx \end{aligned}$$

By Lebesgue's dominated convergence theorem we get

$$\lim_{n \to \infty} f(u_n) = 0 \qquad \text{and thus that} \qquad \inf_{u \in M} f(u) = 0.$$

(III) Conclusion:

Thus, the continuous functional f cannot admit its infimum on M. This shows that M is not compact, despite being bounded and closed.

□

Proposition 267.

<u>*Hyp*</u> *Consider the closed unit ball*

$$B := \{u \in X \ : \ \|u\| \leq 1\}$$

in a normed space $(X, \|\cdot\|)$.

<u>*Concl*</u> *Then*

$$B \text{ is compact} \iff \dim X < +\infty.$$

7.5.4. Theorem of Arzelà-Ascoli

In infinite dimensional spaces, it is difficult to decide whether or not a given set is compact. Thus, every result that establishes the compactness of a set is of main importance. As an example, we mention the following one:

Theorem of Arzelà-Ascoli

Proposition 268.
Consider the space $X = C[a,b]$ *with* $-\infty < a < b < +\infty$ *equipped with the (standard) norm*

$$\|u\|_\infty := \max_{a \le x \le b} |u(x)|.$$

Suppose that $M \subset C[a,b]$ *is a subset that is* uniformly bounded *and* equi-continuous, *i.e. suppose that* M *is uniformly bounded:*

$$\exists R > 0 \text{ such that } \|u\|_\infty \le R, \quad \forall u \in M$$

i.e.

$$\exists R > 0 \text{ such that } |u(x)| \le R, \quad \forall x \in [a,b], \forall u \in M$$

and that M *is equi-continuous:*

$$\begin{aligned} &\forall \varepsilon > 0 \\ &\exists \delta = \delta(\varepsilon) > 0 \text{ such that} \\ &\quad |u(x_1) - u(x_2)| < \varepsilon, \qquad \forall x_1, x_1 \in [a,b] \text{ with } |x_1 - x_2| < \delta \\ &\qquad\qquad\qquad\qquad\qquad\quad\;\; \forall u \in M \end{aligned}$$

Then, the set M *is relatively compact in* $(C[a,b], \|\cdot\|_\infty)$.

7.6. Compact operators

Let X and Y be two normed spaces, and consider an operator

$$A : X \to Y, \quad u \mapsto Au.$$

The following situation occurs frequently:
Let M be a *bounded* subset in X, and let $\{u_n\}_{n=1}^{+\infty}$ be a sequence in M.
Does there exist a sub-sequence $\{u_{n_k}\}_{k=1}^{\infty}$ such that

$$\text{the limit } \lim_{k\to\infty} Au_{n_k} \text{ exists?}$$

Remark that such a sub-sequence exists if

- M is compact, so that a convergent sub-sequence $\{u_{n_k}\}_{k=1}^{\infty}$ exists with $\lim_{k\to\infty} u_{n_k} := u \in M$, and if
- A is continuous, so that $\{Au_{n_k}\}_{k=1}\infty$ is convergent.

Another possibility to achieve this, is to have a so-called *compact* operator.

Definition 269.

Given: An operator

$$A : M \subset X \to Y, \quad u \mapsto Au, \qquad X \text{ and } Y \text{ being normed spaces}$$

we say: A is a *compact* operator iff:

1. this operator A is continuous and such that
2. any *bounded* subset $B \subset M$ is mapped into a *relatively compact* set $A(B) \subset Y$.

Remark that the second condition means that, any bounded sequence $\{u_n\}_{n=1}^{+\infty}$ in X has a sub-sequence $\{u_{n_k}\}_{k=1}^{+\infty}$ such that the limit

$$\lim_{k\to\infty} Au_{n_k}$$

exists in Y.

As a standard example of a compact operator, we mention the integral operators:

Integral operators are compact operators

Example 270.

Le us consider a given, continuous function

$$f : [a,b] \times [a,b] \times [-r,r] \to \mathbb{R}, (x,y,u) \mapsto f(x,y,u)$$

where $-\infty < a < b < +\infty$ and $r > 0$ are fixed.

This function defines an operator

$$A : M \subset C[a,b] \to C[a,b], u \mapsto Au$$

with

$$M := \{u \in C[a,b] \ : \ \|u\|_\infty \leq r\}$$

and

$$(Au)(x) := \int_a^b f(x,\xi,u(\xi))\, d\xi.$$

Remark first of all that A is well-defined, since

$$u \in M \Longrightarrow Au \in C[a,b].$$

Moreover, A is continuous.
Indeed, f is uniformly continuous. This means that

$$\begin{array}{l} \forall \varepsilon > 0 \\ \exists \delta > 0 \text{ such that} \\ \quad \left. \begin{array}{l} u, v \in [-r, r] \\ |u - v| < \delta \end{array} \right\} \implies |f(x, y, u) - f(x, y, v)| < \frac{\varepsilon}{b-a}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall x, y \in [a, b]. \end{array}$$

Thus, if one considers two points u and $v \in M$ with

$$\|u - v\|_\infty = \max_{a \le x \le b} |u(x) - v(x)| < \delta,$$

then

$$|f(x, \xi, u(\xi) - f(x, \xi, v(\xi))| < \frac{\varepsilon}{b - a}, \quad \forall x, \xi \in [a, b].$$

Thus, $u, v \in M$ with $\|u - v\|_\infty < \delta$ implies that

$$\begin{aligned} \|Au - Av\|_\infty &= \max_{a \le x \le b} |(Au)(x) - (Av)(x)| \\ &= \max_{a \le x \le b} \left| \int f(x, \xi, u(\xi)) \, d\xi - \int f(x, \xi, u(\xi)) \, d\xi \right| \\ &\le \max_{a \le x \le b} \int_a^b |f(x, \xi, u(\xi)) - f(x, \xi, v(\xi))| \, d\xi \\ &\le (b - a) \cdot \frac{\varepsilon}{b - a} = \varepsilon. \end{aligned}$$

This show that A is continuous (on M).
But A is not only continuous, it is even compact.
This can be shown in two steps:

1. First we show that $A(M)$ is uniformly bounded in $C[a, b]$;
2. The we show that $A(M)$ is equi-continuous.

Once this is established, we can conclude by Arzelá-Ascoli that $A(M)$ is relatively compact. Hence

$$B \subset M \text{ bounded} \implies A(B) \subset A(M) \text{ is relatively bounded.}$$

In order to show that $A(M)$ is uniformly bounded, we put

$$K := \max_{a \le x, y \le b, -r \le u \le r} |f(x, y, u)| \qquad (\in [0, +\infty[)$$

and we remark that, for all $u \in M$,

$$\begin{aligned} \|Au\|_\infty &= \max_{a\le x\le b} |(Au)(x)| = \max_{a\le x\le b} \left| \int_a^b f(x,\xi,u(\xi))\, d\xi \right| \\ &\le \max_{a\le x\le b} \int_a^b |f(x,\xi,u(\xi)|\ d\xi \le \max_{a\le x\le b} \int_a^b K\, d\xi \\ &= K(b-a) < +\infty. \end{aligned}$$

In order to show that the set $A(M)$ is equi-continuous, we remark that the function f is uniformly continuous. This means that

$$\begin{array}{l} \forall \varepsilon > 0 \\ \exists \delta = \delta(\varepsilon) > 0 \text{ such that} \\ \quad \left. \begin{array}{l} x_1, x_2 \in [a,b] \\ |x_2 - x_1| < \delta \end{array} \right\} \quad \Longrightarrow \quad |f(x_1,y,u) - f(x_2,y,u)| < \frac{\varepsilon}{b-a} \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall y \in [a,b] \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall u \in [-r,r]. \end{array}$$

For such values of x_1 and x_2, we have, $\forall u \in M$,

$$\begin{aligned} |(Au)(x_1) - (Au)(x_2)| &= \left| \int_a^b f(x_1,\xi,u(\xi))\, d\xi - f(x_2,\xi,u(\xi))\, d\xi \right| \\ &\le \int_a^b |f(x_1,\xi,u(\xi))\, d\xi - f(x_2,\xi,u(\xi))|\ d\xi \\ &< \int_a^b \frac{\varepsilon}{b-a}\, d\xi = \varepsilon. \end{aligned}$$

This shows that the set $A(M)$ is equi-continuous.

So we are done:

$$A : M \subset C[a,b] \to C[a,b] \qquad \text{is a compact operator.}$$

Compact operators are of a main interest, since they can be approximated by "finite-range" operators.

Thus, a lot of properties can be expanded from continuous operators operating on finite dimensional spaces to compact operators.

Let us formulate the above mentioned approximation:

Proposition 271.

<u>*Hyp*</u> *Suppose that*

$$A : M \subset X \to Y, \quad u \mapsto Au$$

is a compact operator, where

- X *and* Y *are Banach-spaces over* $\mathbb{K}$ *and*
- $M \subset X$ *is bounded and such that* $M \neq \varnothing$.

<u>*Concl*</u> *For* $n = 1, 2, 3, \ldots$, *there exist continuous operators*

$$A_n : M \subset X \to Y$$

such that

1. $\sup_{u \in M} \|Au - A_n u\| \leq \frac{1}{n}$, *i.e. the approximation is uniformly "good" on* M*;*
2. $\dim \operatorname{span}(A_n(M)) < \infty$, *i.e.* A_n *is an operator with a finite-dimensional range;*
3. $A_n(M)$ *is contained in the smallest convex set of* Y *containing* $A(M)$.

7.7. Brower fixed point theorem

We give now a second fixed point theorem, where the operator no longer must be k-coercive as in Banach's fixed point theorem.

Brower fixed point theorem

Proposition 272.

Hyp *Consider a* continuous *operator*

$$A : M \to M,$$

where $M \neq \varnothing$ *is a*

- *bounded,*
- *closed and*
- *convex*

subset of a finite-dimensional *normed space* X.
(Thus M *is a compact set!)*

Concl *The operator* A *has (at least) one fixed point* $\bar{u} \in M$*:*

$$\exists \bar{u} \in M \text{ such that } A\bar{u} = \bar{u}.$$

Proof. (Proof for the special case where $M \subset \mathbb{R}^2$ is a disc)

Suppose on the contrary that no such $\bar{u} \in M$ exists:

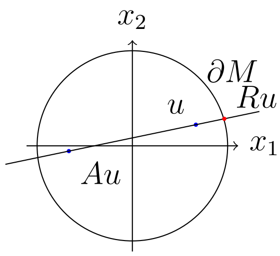
$$Au \neq u, \qquad \forall u \in M.$$

Then we can construct a *continuous* mapping

$$R : M \to \partial M, \qquad \text{(where } \partial M \text{ is a circle in } \mathbb{R}^2\text{)}$$

that keeps every point on the border ∂M fixed.

The line trough u and Au intersects ∂M in two points; choose as Ru the intersection point that is closer to u than to Au.



Remark that

- if $u \in \partial M$, then $Ru = u$ and
- Ru depends in a continuous way on u, since A is continuous.

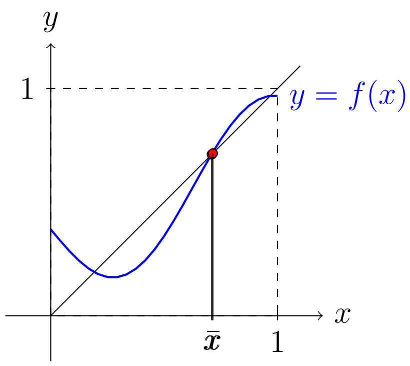
Intuitively, such a continuous mapping where the points on ∂M remain fixed cannot exit. And this can be proven in a strict way with the help of index theory. □

Example 273.
Any *continuous* function

$$f : [0,1] \to [0,1], \quad x \mapsto f(x)$$

has (at least) a fixed point:

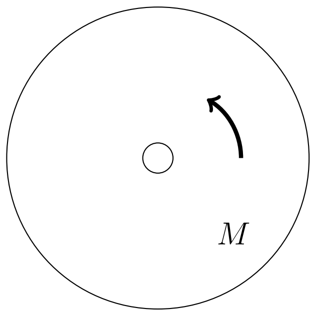
$$\exists \bar{x} \in [0,1] \text{ such that} f(\bar{x}) = \bar{x}.$$



Example 274.
As a counterexample, consider the closed annulus $M \subset \mathbb{R}^2$ and take as a mapping

$$f : M \to M(x,y) \mapsto (x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi)$$

(with $\varphi \neq 0 \mod 2\pi$).
Then M not convex, and the mapping f has no fixed point.



7.8. Schauder fixed point theorem

7.8.1. The fixed point theorem

Brower's fixed point theorem applies to operators acting in finite-dimensional spaces.

This result can now be extended to *compact* operators acting on infinite-dimensional spaces. The proof uses the approximation of a compact operator by finite-dimensional range operators (see above).

Schauder's fixed point theorem

Proposition 275.

Hyp *Consider a* compact *operator*

$$A : M \to M,$$

where $M \neq \varnothing$ *is a*

- *bounded,*
- *closed and*
- *convex*

subset of a Banach space X. *(Thus* M *may no longer be compact!)*

Concl *The operator* A *has (at least) one fixed point* $\bar{u} \in M$*:*

$$\exists \bar{u} \in M \text{ such that } A\bar{u} = \bar{u}.$$

7.8.2. An application: Peano's theorem for ODE

We are going to apply Schauder's fixed point theorem to the following initial value problem:

Find $u(t)$ such that

$$\begin{cases} \dot{u}(t) = f(t, u(t)), & \text{for } t \in [0, b] \qquad \text{with } b > \\ u(0) = u_0, \end{cases}$$

where

- $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a given *continuous* function and where
- $u_0 \in \mathbb{R}$ is a given initial value.

This initial value is equivalent to the following one:

Find $u \in C[0,b]$ such that

$$u(t) = u_0 + \int_0^t f(\tau, u(\tau))\, d\tau, \qquad \forall t \in [0,b].$$

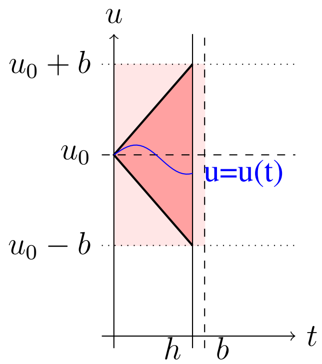
We give now an "abstract" formulation of this problem as a fixed point problem.
We put

$$K := \max_{\substack{0 \le t \le b \\ -b \le u - u_0 \le b}} |f(t,u)|$$

and

$$h := \min\left\{b, \frac{b}{K}\right\}$$

(so that $h \cdot K \le b$).



We can give now the following formulation (as an "abstract" fixed point problem) to our initial value problem:

Find $u \in C[0,b]$ such that

$$u(t) = (Au)(t), \qquad \forall t \in [0,h],$$

where

$$A : M \to M, \qquad (Au)(t) = u_0 + \int_0^t f(\tau, u(\tau))\, d\tau$$

and

$$M = \{u \in C[0,h] \ : \ \max_{0 \le t \le h} |u(t) - u_0| \le b\}.$$

Remark that

- The subset M is closed, convex, bounded and non-empty.

- The operator A maps the set M on itself:

$$u(\cdot) \in M \Longrightarrow (Au)(\cdot) \in M.$$

Indeed

$$\begin{aligned} |(Au)(t) - u_o| &= \left| \int_0^t f(\tau, u(\tau))\, d\tau \right| \\ &\leq \int_0^t |f(\tau, u(\tau))|\, d\tau \\ &\leq \int_0^t K\, d\tau = K \cdot t \\ &\leq K \cdot h \leq b. \end{aligned}$$

- A, as an integral operator with a continuous kernel, is a compact operator.

Hence we may apply Schauder's fixed point theorem to show that there exists (at least) one fixed point $\bar{u}$:

$$\bar{u} \in M, \quad A\bar{u} = \bar{u}.$$

This fixed point $\bar{u}$ is a solution of our initial value problem.
Thus we get

Peano's theorem for ODE

Proposition 276.

<u>*Hyp*</u> *Suppose given an initial condition $u_o \in \mathbb{R}$ and a* continuous *function*

$$f : [0, b] \times [u_0 - b, u_0 + b] \to \mathbb{R}, (t, u) \mapsto f(, u).$$

Choose $h \in]0, b]$ in such a way that

$$h \cdot \max_{\substack{0 \leq t \leq b \\ u_o - b \leq u \leq u_0 + b}} |f(t, u)| \leq b.$$

<u>*Concl*</u> *The initial value problem:*

Find $u(t)$ such that

$$\begin{cases} \dot{u}(t) = f(t, u(t)), & \textit{for } t \in [0, h] \\ u(0) = u_0, \end{cases}$$

has (at least) one solution.

8

Linear Operators

8.1. Boundedness and continuity of linear oparators

We discuss in this section an important class of operators

$$A : L \subset X \to Y, \qquad u \mapsto Au$$

where X and Y are linear spaces. Such operators occur for example in the description of filters and whenever the "principle of superposition" holds.

Definition 277.
Given: linear spaces X and Y and an operator

$$A : L \subset X \to Y, \qquad u \mapsto Au$$

we say: A to be a *linear operator* iff:

1. its domain L is a (non-empty) linear space (f.ex. $L = X$) and if
2. the principle of superposition holds:

$$A(\alpha u + v) = \alpha Au + Av, \qquad \forall u, v \in L, \forall \alpha \in \mathbb{K}.$$

Remark 278. *As for any operator, on may consider its range:*

$$\mathrm{Ran}(A) := \{v \in Y \ : \ v = Au \textit{ for some } u \in L\},$$

i.e.

$$\mathrm{Ran}(A) = A(L).$$

If the operator A is linear, its range $\mathrm{Ran}(A)$ is a linear sub-space of Y.

Remark 279. *For linear operator, the linear sup-space*

$$\ker(A) := \{u \in L \ : \ Au = 0\} = A^{-1}(0)$$

is of main interest: it is called the kernel of A.
As a linear sub-space of X, one has

$$\{0\} \subset \ker(A).$$

If

$$\ker(A) \neq \{0\},$$

we say that the kernel is not trivial.

Let us mention the following property of linear operators A:

$$A \text{ is an injection} \iff \ker(A) = \{0\}$$

that relays on the fact that $Au = Av$ is equivalent to $A(u - v) = 0$.

Example 280.

If X and Y are finite-dimensional linear spaces, linear mappings $A : X \to Y$ are given by matrices: every linear mapping corresponds to such a matrix (in a unique way) and vice versa every such matrix corresponds (in a unique way) to a linear mapping as soon as on introduces bases in X and Y:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{pmatrix}$$

where $n = \dim Y$ and $m = \dim X$.

If $e_1, \dots e_m$ is a basis in X and $f_1, \dots f_n$ is a basis in Y, the

$$Ae_j = \sum_{i=1}^{n} a_{ij} f_i, \qquad j = 1, 2, \dots, m.$$

Moreover, if

$$x = \sum_{j=1}^{m} \xi_j e_j \qquad \text{and} y = \sum_{i=1}^{n} \eta_i f_i,$$

then $Ax = y$ means

$$\eta_i = \sum_{j=1}^{m} a_{ij} \xi_j, \qquad (i = 1, 2, \dots, n).$$

Thus we get

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$$

One easily sees now, that linear operators acting on finite-dimensional spaces are continuous: the result Ax changes as little as you want at the price that you make the changes in the input x small enough.
For linear operators acting on infinite-dimensional spaces the is no longer the case!

Proposition 281.

Hyp *Consider a linear operator*

$$A : X \to Y, \quad u \mapsto Au$$

where X and Y are normed spaces over $\mathbb{K}$.

Concl *Then this operator A is continuous if and only if this operator is bounded. Thereby we say that the operator A is bounded if*

$$\exists c \in [0, +\infty[\text{ with } \|Au\|_Y \leq c \cdot \|u\|_X, \qquad \forall u \in X.$$

In short

$$A \text{ continuous} \Longleftrightarrow A \text{ bounded}.$$

We will give a proof of this proposition. But before doing so, we introduce an notation.

Definition 282.

Given: a bounded operator $A : X \to Y$
we define: the *norm of the operator A* as:

$$\|A\| := \sup_{u \neq 0} \frac{\|Au\|_Y}{\|u\|_X}$$

(maximal stretching factor!).

Thus we have

$$\boxed{\|Au\|_Y \leq \|A\| \cdot \|u\|_X, \qquad \forall u \in X.}$$

For $u \neq 0$ and $\alpha \neq 0$, we have

$$\frac{\|A(\alpha u)\|_Y}{\|\alpha u\|_X} = \frac{\|\alpha(Au)\|_Y}{\|\alpha u\|_X} = \frac{|\alpha| \cdot \|Au\|_Y}{|\alpha| \cdot \|u\|_X} = \frac{\|Au\|_Y}{\|u\|_X}$$

Thus

$$\boxed{\|A\| = \sup_{\|u\|_X = 1} \|Au\|_Y = \sup_{\|u\|_X \leq 1} \|Au\|_Y = \sup_{u \neq 0} \frac{\|Au\|_Y}{\|u\|_X}}$$

We give now the proof of the above proposition.

Proof of Proposition 281. **(I)** A **continuous** $\Longrightarrow$ A **bounded:**
Assume on the contrary that the continuous, linear operator A is not bounded. Thus, we may find a sequence $\{u_n\}_{n=1}^{+\infty}$ is X with

$$\|Au_n\|_Y \geq n \cdot \|u_n\|_X.$$

We remark that $u_n \neq 0$, since $A0 = 0$. Thus we may consider the sequence $\{w_n\}_{n=1}^{+\infty}$ in X defined by

$$w_n := \frac{1}{n} \frac{u_n}{\|u_n\|_X}.$$

Remark that

- We have
$$\lim_{n\to\infty} \|w_n\|_X = 0.$$
Indeed
$$\|w_n\|_X = \frac{1}{n} \frac{\|u_n\|_X}{\|u_n\|_X} = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$
- Thus, since A is continuous
$$\lim_{n\to\infty} Aw_n = 0.$$
- However, this is in (a desired) contradiction with
$$\begin{aligned} \|Aw_n\|_Y &= \left\| A\left(\frac{1}{n} \frac{u_n}{\|u_n\|_X}\right) \right\|_Y = \frac{1}{n \cdot \|u_n\|_X} \cdot \|Au_n\|_Y \\ &\geq \frac{1}{n \cdot \|u_n\|_X} \cdot n \cdot \|u_n\|_X = 1. \end{aligned}$$

(II)A **bounded** $\Longrightarrow$ A **continuous:**
Suppose now that A is a bounded, linear operator:

$$\exists c \in [0, +\infty[\text{ with } \|Au\|_Y \leq c \cdot \|u\|_X, \qquad \forall u \in X.$$

For a given $\varepsilon > 0$, we put $\delta := \frac{\varepsilon}{c}$. Then

$$\|u - v\|_X < \delta$$

implies

$$\|Au - Av\|_Y = \|A(u - v)\|_Y \leq c \cdot \|u - v\|_X < c \cdot \delta = \varepsilon.$$

Thus A is continuous (and even *uniformly continuous*).

□

One can considered integral operators of the form

$$(Au)(x) = \int_a^b f(x, \xi, u(\xi))\, d\xi.$$

Le consider now a special case of such operators where

$$f(x, y, u) = k(x, y)u.$$

Proposition 283.

<u>Hyp</u> *Given a* continuous *function*

$$k : [a, b] \times [a, b] \to \mathbb{R},$$

where $-\infty < a < b < +\infty$*, we consider the operator*

$$A : (C[a, b], \|\cdot\|_\infty) \to (C[a, b], \|\cdot\|_\infty) \quad u \mapsto Au$$

given by

$$(Au)(x) := \int_a^b k(x, y)u(y)\, dy, \quad x \in [a, b].$$

<u>Concl</u> *Beside being a compact operator,* A *is a linear, bounded operator with*

$$\|A\| \le (b - a) \cdot \max_{a \le x, y \le b} |k(x, y)|.$$

Proof. Clearly, A is a linear operator. Indeed, for u, $v \in C[a, b]$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} A(\alpha u + v)(x) &= \int_a^b k(x.y) \cdot [\alpha \cdot u(y) + v(y)]\; dy \\ &= \alpha \int_a^b k(x, y)u(y)\, dy + \int_a^b k(x, y)v(y)\, dy \\ &= \alpha(Au)(x) + (Av)(x), \qquad \forall x \in [a, b] \end{aligned}$$

i.e.

$$A(\alpha u + v) = \alpha Au + Av.$$

Moreover, we have

$$\begin{aligned} |(Au)(x)| &= \left|\int_a^b k(x,y)u(y)\ dy\right| \\ &\leq \int_a^b |k(x,y)u(y)|\ dy = \int_a^b |k(x,y)|\cdot|u(y)|\ dy \\ &\leq \max_{a\leq x,y\leq b}|k(x,y)|\cdot\max_{a\leq y\leq b}|u(y)|\cdot(b-a) \end{aligned}$$

so that

$$\|Au\|_\infty \leq (b-a)\cdot\max_{a\leq x,y\leq b}|k(x,y)|\cdot\|u\|_\infty.$$

This shows that

$$\|A\| \leq (b-a)\cdot\max_{a\leq x,y\leq b}|k(x,y)|.$$

□

8.2. The space of bounded, linear operators

Definition 284.

Given: normed spaces X and Y (over $\mathbb{K}$)

we define: the space of bounded, linear operators $L(X,Y)$ as:

as the set given by

$$L(X,Y) := \{A : X \to Y \ :\ A \text{ is linear and bounded}\}$$

This space $L(X,Y)$ can equipped with

- an addition through

$$(A+B)u := Au + Bu, \qquad (A,B\in L(X,Y))$$

 and

- a multiplication by scalars through

$$(\alpha A)u = \alpha(Au), \qquad (A\in L(X,Y), \alpha\in\mathbb{K}).$$

Proposition 285.
If X and Y are normed spaces over $\mathbb{K}$, the space

$$(L(X,Y),+,\cdot)$$

is a vector space over $\mathbb{K}$.

For every $A \in L(X,Y)$, we have set

$$\|A\| := \sup_{\|u\|_X=1} \|Au\|_Y.$$

The following proposition says that this is indeed an norm on $L(X,Y)$.

Proposition 286.
If X and Y are normed spaces over $\mathbb{K}$, the space

$$(L(X,Y),+,\cdot) \quad \textit{equipped with } \|A\| := \sup_{\|u\|_X=1} \|Au\|_Y$$

is a normed space over $\mathbb{K}$.

Proof. It is enough to show that $\|A\| := \sup_{\|u\|_X=1} \|Au\|_Y$ defines a norm on $L(X,Y)$.

(I) $\|\cdot\|$ is strictly positive:
Indeed

- $\|A\| = \sup_{\|u\|_X=1} \|Au\|_Y \geq 0, \forall A \in L(X,Y)$;
- Moreover,

$$\|A\| = \sup_{\|u\|_X=1} \|Au\|_Y = 0 \Longleftrightarrow Au = 0, \forall u \in X \Longleftrightarrow A = 0.$$

(II) $\|\cdot\|$ is homogeneous:
Indeed, $\forall \alpha \in \mathbb{K}$,

$$\|\alpha A\| = \sup_{\|u\|_X=1} \|\alpha(Au)\|_Y = |\alpha| \cdot \sup_{\|u\|_X=1} \|Au\|_Y = |\alpha| \cdot \|A\|.$$

(II) Triangular inequality:
For all A and $B \in L(X,Y)$, we have

$$\begin{aligned}
\|A+B\| &= \sup_{\|u\|_X=1} \|(A+B)u\|_Y = \sup_{\|u\|_X=1} \|Au+Bu\|_Y \\
&\leq \sup_{\|u\|_X=1} (\|Au\|_Y + \|Bu\|_Y) \\
&\leq \sup_{\|u\|_X=1} \|Au\|_Y + \sup_{\|u\|_X=1} \|Bu\|_Y \\
&= \|A\| + \|B\|.
\end{aligned}$$

□

Proposition 287.

<u>*Hyp*</u>

- *X is a normed space over $\mathbb{K}$ (for example a Banach space) and*
- *Y is a Banach space over $\mathbb{K}$,*

<u>*Concl*</u> *The space*

$$(L(X,Y),+,\cdot) \quad \text{equipped with } \|A\| := \sup_{\|u\|_X=1} \|Au\|_Y$$

is a Banach space over $\mathbb{K}$.

Proof. We must show that the space $L(X,Y)$ is complete as soon as the target space Y is complete.

So let us consider a Cauchy sequence $\{A_n\}_{n=1}^{+\infty}$ in $L(X,Y)$:

$$\begin{aligned} &\forall \varepsilon > 0 \\ &\exists n_0 = n_0(\varepsilon) \text{ such that} \\ &\qquad n, m \geq n_0 \Longrightarrow \|A_n - A_m\| < \epsilon \end{aligned}$$

But

$$\|A_n u - A_m u\|_Y = \|(A_n - A_m)u\|_Y \leq \|A_n - A_m\| \cdot \|u\|_X$$

implies that

$$\begin{aligned} &\forall u \in X \\ &\qquad \{A_n u\}_{n=1}^{\infty} \text{ is a Cauchy sequence in } Y. \end{aligned}$$

But Y is Banach, so the Cauchy sequence $\{A_n u\}_{n=1}^{\infty}$ converges in Y, and we may put

$$Au := \lim_{n\to\infty} A_n u, \qquad \forall u \in X.$$

In this way, we get an operator $A : X \to Y$. We show that

- A is a linear operator;
- A is a bounded operator and
- $\lim_{n\to\infty} \|A - A_n\| = 0$, i.e. $\lim_{n\to\infty} A_n = A$ in $(L(X,Y), \|\cdot\|)$.

(I) A is a linear operator:

Indeed, $\forall \alpha \in \mathbb{K}$, $\forall u, v \in X$, we have

$$\begin{aligned} A(\alpha u + v) &= \lim_{n\to\infty} \underbrace{A_n(\alpha u + v)}_{=\alpha A_n u + A_n v} \\ &= \alpha \cdot \lim_{n\to\infty} A_n u + \lim_{n\to\infty} A_n v \\ &= \alpha A u + A v. \end{aligned}$$

(II) A is a bounded operator, so that $A \in L(X,Y)$:
Indeed, for $n \geq n_0$,

$$\begin{aligned} \|A_n u\|_Y &= \|A_n u - A_{n_0} u + A_{n_0} u\|_Y = \|(A_m - A_{n_0})u + A_{n_0} u\|_Y \\ &\leq \|(A_n - A_{n_0})u\|_Y + \|A_{n_0} u\|_Y \\ &\leq \|A_n - A_{n_0}\| \cdot \|u\|_X + \|A_{n_0}\| \cdot \|u\|_X \\ &= = \left[\|A_n - A_{n_0}\| + \|A_{n_0}\|\right] \cdot \|u\|_X \end{aligned}$$

i.e., if $n_0 = n_0(\varepsilon)$ is large enough,

$$\|Au\|_Y = \lim_{n\to\infty} \|A_n u\|_y \leq [\varepsilon + \|A\|_{n_0}] \cdot \|u\|_X$$

so that

$$\|A\| \leq \varepsilon + \|A\|_{n_0} < +\infty.$$

Thus $A \in L(X,Y)$.
(III) $\lim_{n\to\infty} \|A - A_n\| = 0$
Indeed, for n, $m \geq n_0$, we have

$$\begin{aligned} \|A_n u - A_m u\|_Y &\leq \|A_n - A_m\| \cdot \|u\|_X < \varepsilon \cdot \|u\|_X, \qquad \forall u \in X \\ \|Au - A_m u\|_Y &= \lim_{n\to\infty} \|A_n u - A_m u\|_Y \leq \varepsilon \cdot \|u\|_X, \qquad \forall u \in X \end{aligned}$$

i.e.

$$\|A - A_m\| \leq \varepsilon, \qquad \forall m \geq n_0.$$

Thus

$$\lim_{n\to\infty} \|A - A_n\| = 0, \quad \text{i.e.} \quad \lim_{n\to\infty} A_n = A.$$

□

8.3. The dual space

Definition 288.
Let X be a normed space over $\mathbb{K}$.

1. f is *linear, continuous functional*:

   f is a linear and bounded operator

   $$f : X \to \mathbb{K}$$

   i.e. an element of the Banach space $L(X, \mathbb{K})$.

2. the *dual space of X*:

The collection $L(X, \mathbb{K})$ of all linear, continuous functionals.

The dual space of X is denoted by

$$X' := L(X, \mathbb{K}).$$

Moreover, we introduce the following notation:

$$\boxed{\langle f, u \rangle := f(u), \qquad \forall u \in X, \forall f \in X'.}$$

Proposition 289.
If X is a normed space (or even a Banach space), its dual space X' is a Banach space.

Example 290.
Consider the space $\mathbb{R}^p$ $(p \geq 1)$ equipped with the norm

$$\|x\|_2 := \sqrt{\sum_{k=1}^{p} \xi^2}, \qquad \text{for } x = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_p \end{pmatrix}.$$

Then, its dual space is given by the space of $1 \times p$-matrices:

$$(\mathbb{R}^p)' := \{y = \begin{pmatrix} \eta_1 & \dots & \eta_p \end{pmatrix} \; : \; \eta_k \in \mathbb{R} \quad \text{, for } k = 1, 2, \dots, p\}$$

equipped with the norm

$$\|y\|_2 = \sqrt{\sum_{k=1}^{p} \eta^2}$$

Remark that

$$\langle y, x \rangle = \sum_{k=1}^{p} \eta_k \xi_k.$$

Remark 291. *In the above example, the normed space could be identified with itself. This is, however,* not *true in general.*

Example 292.
Consider the Banach space

$$C[a, b], \qquad \text{equipped with the norm } \|u\|_\infty := \max_{a \leq x \leq b} |x(x)|,$$

where $-\infty < a < b < +\infty$.

Let us show that any fixed element $v \in C[a,b]$ can be viewed as an element of the dual space $(C[a,b])'$.

Thus we may write
$$C[a,b] \subset (C[a,b])'.$$

So let $w \in C[a,b]$ be fixed. The we define an operator $f_w : C[a,b] \to \mathbb{R}$ through

$$f_w(u) := \int_a^b w(x)u(x)\, dx$$

This operator f_w is linear, since, for $\alpha \in \mathbb{R}$ and $u, v \in C[a,b]$,

$$\begin{aligned} f_w(\alpha u + v) &= \int_a^b w(x)\left[\alpha u(x) + v(x)\right]\, dx \\ &= \alpha \int_a^b w(x)u(x)\, dx + \int_a^b w(x)v(x)\, dx \\ &= \alpha f_w(u) + f_w(v). \end{aligned}$$

Moreover, f_w is bounded, since

$$\begin{aligned} |f_w(u)| &= \left|\int_a^b w(x)u(x)\, dx\right| \le \int_a^b |w(x)| \cdot |u(x)|\, dx \\ &\le \|w\|_\infty \cdot \|u\|_\infty \cdot (b-a) \end{aligned}$$

gives
$$\|f_w\| \le (b-a) \cdot \|w\|_\infty.$$

Remark however that
$$C[a,b] \ne (C[a,b])'$$

since for example the following linear, bounded functional cannot be represented by a continuous function w:
$$\delta(u) := u(a), \qquad u \in C[a,b].$$

Thus
$$\boxed{C[a,b] \subsetneqq (C[a,b])'}$$

8.4. Operational calculus

Remark 293. *If $A \in L(X,Y)$ and $B \in L(Y,Z)$, the we denote by*

$$BA : X \to Y, \quad u \mapsto B(Au))$$

the composition of these mappings. Remark that

$$\|B(Au)\|_Z \leq \|B\| \cdot \|Au\|_Y \leq \|B\| \cdot \|A\| \cdot \|u\|_X,$$

so that

- $BA \in L(X; Z)$ *and*
- $\|BA\| \leq \|B\| \cdot \|A\|$.

Remark moreover that composition is continuous:

$$\left.\begin{array}{l} A_n \to A \text{ in } L(X,Y) \\ B_n \to B \text{ in } L(Y,Z) \end{array}\right\} \Longrightarrow B_n A_n \to BA \text{ in } L(X,Z).$$

Remark 294. *If A and $B \in L(X; X)$, the*

- $BA \in L(X; X)$ *and*
- $\|BA\| \leq \|B\| \cdot \|A\|$.

In particular

$$\|A^k\| \leq \|A\|^k, \qquad \text{for } k = 0, 1, 2, \ldots.$$

Proposition 295.

<u>*Hyp*</u> *Suppose given a power series*

$$F(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_k \in \mathbb{K}$$

with

$$\sum_{k=0}^{\infty} |a_k| \cdot |z|^k < +\infty \qquad \text{for } |z| < r,$$

where $r > 0$ is the radius of convergence.
Let X be a Banach space over $\mathbb{K}$.

Concl For any

$$A \in L(X,X) \qquad \text{with } \|A\| < r$$

the series

$$F(A) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k A^k$$

converges in $L(X,X)$ and $F(A) \in L(X,X)$.
Remark thereby that

$$A^0 = \mathbb{I} \qquad \text{and} \qquad A^{k+1} = A^k A.$$

Definition 296.
Given: a Banach space X
we define: $L_{\text{inv}}(X,X)$ as:
the following subset of Banach space $L(X,X)$:

$$L_{\text{inv}}(X,X); = \{A \in L(X,X) \ : \ \exists A^{-1} \in L(X,X)\}.$$

Thereby, we denote by A^{-1} the inverse operator of A:

$$A^{-1}A = AA^{-1} = \mathbb{I}.$$

Remark 297. *Thus, the operator A belongs to $L_{inv}(X,X)$ if and only if*

1. $A \in L(X,X)$;
2. $A : X \to X$ *is a bijection, so that the inverse* $A^{-1} : X \to X$ *exists;*
3. A^{-1} *is linear and bounded, too.*

Let us consider the geometric series

$$\frac{1}{1-z} := \sum_{k=0}^{\infty} z^k, \qquad \text{for } |z| < 1.$$

Hence, for all $A \in L(X,X)$ with $\|A\| < 1$, we may put

$$\frac{1}{\mathbb{I} - A} := \sum_{k=0}^{\infty} A^k = \mathbb{I} + A + A^2 + A^3 + \cdots .$$

Proposition 298.

<u>*Hyp*</u> X *be a Banach space over* $\mathbb{K}$.

<u>*Concl*</u> *We have*

$$\forall A \in L(X,X) \quad \textit{with } \|A\| < 1$$
$$\exists (\mathbb{I} - A)^{-1} \in L(X,X) \textit{ and}$$
$$(\mathbb{I} - A)^{-1} = \tfrac{1}{\mathbb{I}-A} = \textstyle\sum_{k=0}^{\infty} A^k = \mathbb{I} + A + A^2 + A^3 + \cdots .$$

In short

$$\boxed{\begin{array}{l}\forall A \in L(X,X),\\ \qquad \|A\| < 1 \Longrightarrow (\mathbb{I} - A) \in L_{inv}(X,X).\end{array}}$$

Proof. Remark that

$$\|A\| < 1 \Longrightarrow \lim_{n\to\infty} \|A\|^{n+1} = 0,$$

so that

$$\lim_{n\to\infty} A^{n+1} = 0.$$

Thus

$$\begin{aligned}(\mathbb{I} - A)\sum_{k=0}^{\infty} A^k &= \lim_{n\to\infty} (\mathbb{I} - A)\sum_{k=0}^{n} A^k \\ &= \lim_{n\to\infty} \left(\sum_{k=0}^{n} A^k - \sum_{k=1}^{n+1} A^k\right) = \lim_{n\to\infty} \left(\mathbb{I} - A^{n+1}\right) = \mathbb{I}.\end{aligned}$$

□

In a similar way, one may consider the exponential series

$$\exp(z) = e^z := \sum_{k=0}^{\infty} \frac{1}{k!} z^k \qquad \text{for } z \in \mathbb{K}.$$

Proposition 299.

<u>*Hyp*</u> X *a Banach space*

<u>Concl</u> *For every $A \in L(X, X)$, one may define the exponential of A through*

$$\exp(A) = e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k \in L(X, X).$$

One has, for $s, t \in \mathbb{K}$,

$$e^{tA} e^{sA} = e^{(t+s)A}.$$

9

Fourier series: the classical approach

9.1. Signals

The general framework

The purpose of signal theory is to study

- signals and
- the systems that transform them.

Signals and their mathematical description

The observation of some phenomenon yields quantities that depend on time. We call them signals.

Their mathematical description is done by functions or their generalization: the distributions.

In this lecture we will only consider *deterministic* signals without including this property explicitly: so all here considered signals will take values with no stochastic element.

An analog signal is a signal where the time is modeled by $\mathbb{R}$.

A discrete signal is a signal where time is discrete (modeled by $\mathbb{Z}$ for example).

A discrete signal frequently results from sampling an analog signal.

We will consider in this lecture real valued as well as complex valued signals.

Thus analog signals are modeled in a first step by

$$x : \mathbb{R} \to \mathbb{R} \qquad \text{or} \qquad x : \mathbb{R} \to \mathbb{C}$$

whereas discrete signals are modeled by

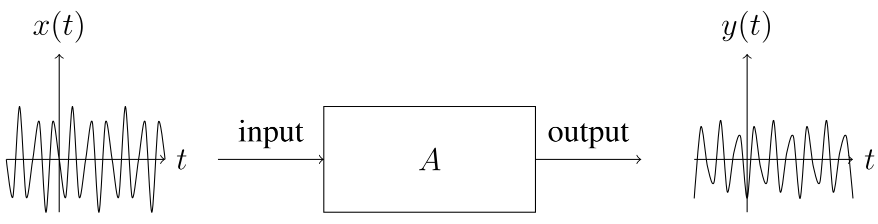
$$x : \mathbb{Z} \to \mathbb{R} \qquad \text{or} \qquad x : \mathbb{Z} \to \mathbb{C}.$$

Systems and their mathematical description

A process, where one can distinguish

- an input signal and
- an output signal

will be called a system.



From a mathematical point of view, systems will be described by:

- A space X consisting of all possible input signals $x(t)$,
- A space Y containing the resulting ouput signals $y(t)$ and
- An operator
$$A : X \to Y, \qquad y \mapsto y = A(x).$$

9.2. Systems

Let us consider a system described by an operator

$$A : X \to Y, \qquad y \mapsto y = A(x),$$

where X and Y are 'function' spaces, i.e. linear $\mathbb{K}$-vector spaces whose elements are functions. Hence elements of these spaces may be added (pointwise addition) and multiplied by a scalar (pointwise multiplication):

$$(x_1 + x_2)(t) = x_1(t) + x_2(t), \qquad \forall t \tag{9.1}$$
$$(\lambda \cdot x)(t) = \lambda \cdot x(t), \qquad \forall t \tag{9.2}$$

where $\lambda \in \mathbb{K}$.

We denote by $\mathbb{K}$ one of the fields $\mathbb{R}$ or $\mathbb{C}$. A scalar is an element of this field.

Linear systems

Definition 300.
Given: A system
$$A : X \to Y, \qquad y \mapsto y = A(x),$$

we say: this system is *linear* iff:

1. $A(x_1 + x_2) = A(x_1) + A(x_2), \forall x_1, x_2 \in X$ and
2. $A(\lambda \cdot x) = \lambda \cdot A(x), \forall x \in X$ and $\forall \lambda \in \mathbb{K}$.

Remark 301. *For linear systems, we usually write Ax instead of $A(x)$*

Remark 302. *Linear systems are systems where the* principle of superposition *holds.*

Time invariant systems

Definition 303.
Given: A system

$$A : X \to Y, \qquad y \mapsto y = A(x),$$

we say: this system is *time invariant* iff:
a translation in time of the input leads to the same translation in time of the output.

Remark 304. *If we denote by τ_a the delay operator defined by*

$$\tau_a x(t) := x(t-a), \qquad \forall t$$

then a system A is time invariant if and only if

$$A\tau_a = \tau_a A, \qquad \forall a \in \mathbb{R}.$$

LTI-systems

Definition 305.
Given: A system

$$A : X \to Y, \qquad y \mapsto y = A(x),$$

we say: this system is *linear, time invariant (LTI)* iff:

1. linear and
2. time invariant.

Causality of systems

Definition 306.

Given: A system

$$A : X \to Y, \qquad y \mapsto y = A(x),$$

we say: this system is *causal* iff:
the response at time t_0 depends only on the input signal before t_0, i.e.

$$x_1(t) = x_2(t), \quad \forall t < t_0 \Longrightarrow A(x_1(t)) = A(x_2(t)), \quad \forall t < t_0.$$

Remark 307. *Causality is a necessary condition for a system to be physically realizable.*

Continuity of systems

When the signals are analog, one uses different kinds of norms in order to measure the 'bigness' of a signal defined on some interval I:

1. *uniform norm:* $\|x\|_\infty := \sup_{t\in I} |x(t)|$;
2. L^1*-norm:* $\|x\|_{L^1} := \int_I |x(t)|\, dt$;
3. L^2*-norm:* $\|x\|_{L^2} := \left[\int_I |x(t)|^2\, dt\right]^{1/2}$.

   Remark that this norm is associated to a 'scalar product':

$$\langle x_1 \mid x_2 \rangle := \int_I x_1(t) \cdot \overline{x_2(t)}\, dt.$$

When the signals are discrete, one uses different kinds of norms in order to measure the 'bigness' of a signal:

1. *uniform norm:* $\|x\|_\infty := \sup_{n\in\mathbb{Z}} |x_n|$;
2. L^1*-norm:* $\|x\|_1 := \sum_{n=-\infty}^{+\infty} |x_n|$;
3. L^2*-norm:* $\|x\|_2 := \left[\sum_{n=-\infty}^{+\infty} |x_n|^2\right]^{1/2}$.

Definition 308.
Given: A (discrete or analog) system

$$A : X \to Y, \qquad y \mapsto y = A(x),$$

we say: this system is *continuous* iff:
the change in the output is as small as you want if the change in the input is accordingly small enough:

$$\|x_n - x\| \to 0 \quad \Longrightarrow \quad \|A(x_n) - A(x)\| \to 0 \text{ as } n \to \infty.$$

Thereby $\|\cdot\|$ is any of the above introduced norms.

9.3. Convolution of signals

Operations with signals

We have yet mentioned, that

- signals can be added and
- signals can be multiplied by constant scalar.

We introduce now an new operation on signals: the convolution.

9.3.1. The convolution of analog signals

Definition 309.
Given: two analog signals x and y
we define: the *convolution $x * y$ of x and y* as:

$$(x * y)(t) := \int_{-\infty}^{+\infty} x(t-\tau) \cdot y(\tau)\, d\tau \qquad (t \in \mathbb{R}).$$

Remark 310. *The above integral must exist for all (or at least, as we will see, for almost all) t.*

One can show that a sufficient condition for this is that

$$\int_{-\infty}^{+\infty} |x(t)|\, dt, \int_{-\infty}^{+\infty} |y(t)|\, dt \quad \textit{both exist in } \mathbb{R},$$

or more generally that

$$x, y \in L^1(\mathbb{R}) \qquad \text{(see below).}$$

Another sufficient condition is that both x and y are piecewise continuous, that one is bounded and the other absolutely integrable over $\mathbb{R}$.

9.3.2. The consolution of periodic signals

Definition 311.
Given: two T-periodic (analog) signals x and y
we define: their *convolution $x * y$* as:

$$(x * y)(t) := \int_0^T x(t-\tau) \cdot y(\tau)\, d\tau \qquad (t \in \mathbb{R}).$$

Remark 312. *The so defined signal $x * y$ is T-periodic, too.*

9.3.3. The convolution of discrete signals

Definition 313.
Given: two sequences $c = \{c_k\}_{k \in \mathbb{Z}}$ and $d = \{d_k\}_{k \in \mathbb{Z}}$
we define: the *convolution $c * d$ of c and d* as:
the sequence $\{(c * d)_k\}_{k \in \mathbb{Z}}$ given by

$$(c * d)_k := \sum_{j=-\infty}^{+\infty} c_{k-j} \cdot d_j \qquad (k \in \mathbb{Z}).$$

Remark 314. *The above sum must converge for all values of $k \in \mathbb{Z}$.*
On can show that this is the case if (f.ex.)

$$\sum_{k=-\infty}^{+\infty} |c_k|, \sum_{k=-\infty}^{+\infty} |d_k| < \infty.$$

Sliding average viewed as convolution

Example 315.

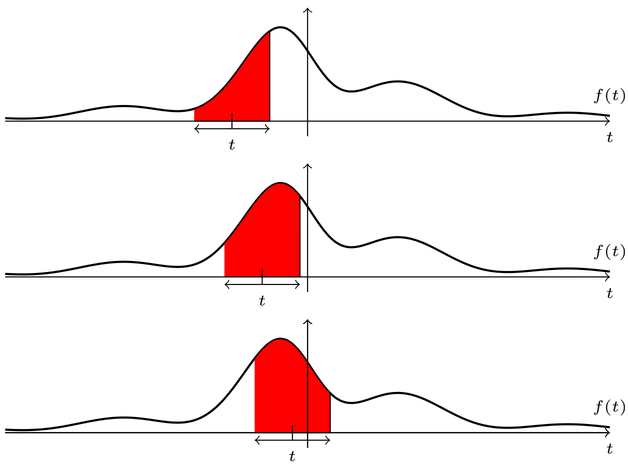
For a fixed value of $h > 0$, we consider the function

$$a_h(t) := \begin{cases} \frac{1}{2h} & \text{, if } -h < t < h \\ 0 & \text{, elsewhere.} \end{cases}$$

Let us compute the convolution of a_h and a piecewise continuous function f:

$$\begin{aligned}(a_h * f)(t) &= \int_{-\infty}^{+\infty} a_h(t-\tau) \cdot f(\tau)\, d\tau \\ &= \frac{1}{2h} \int_{-h<t-\tau<h} f(\tau)\, d\tau \\ &= \frac{1}{2h} \int_{t-h}^{t+h} f(\tau)\, d\tau.\end{aligned}$$

Thus $a_h * f$ is a sliding average:



9.3.4. The sliding strip method to compute convolutions

Example 316.
Let us compute the convolution of

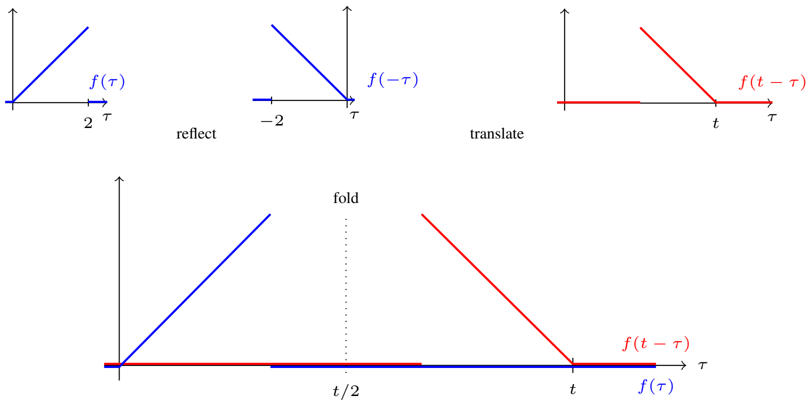
$$f(t) = \begin{cases} t & \text{, for } 0 < t < 2 \\ 0 & \text{, elsewhere} \end{cases} \quad \text{and} \quad g(t) = \begin{cases} 1 & \text{, for } 0 < t < 1 \\ 0 & \text{, elsewhere.} \end{cases}$$

Then

$$(f * g)(t) = \int_{\substack{0<t-\tau<2 \\ 0<\tau<1}} f(t-\tau) \cdot g(\tau)\, d\tau = \int_{\min\{1,\max\{0,t-2\}\}}^{\max\{0,\min\{1,t\}\}} f(t-\tau)\, d\tau$$

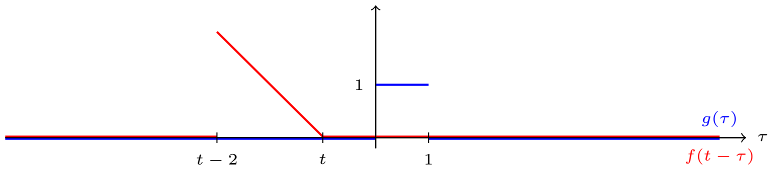
There is a better way to organize the computations!

In a first step, one constructs the graph of $f(t-\tau)$. This can be done either by a reflection followed by a translation or by a folding:



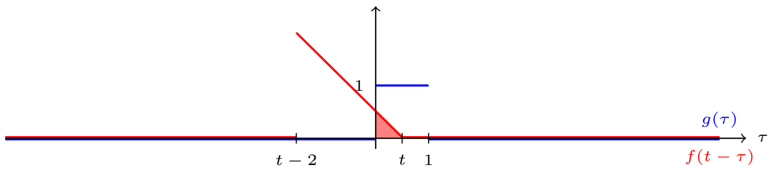
Then we overlap the graphs of $f(t-\tau)$ and $g(\tau)$ step by step:

• If $t \leq 0$:



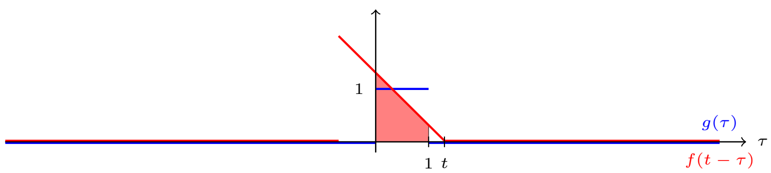
$$(f * g)(t) = 0$$

• If $0 \leq t \leq 1$:



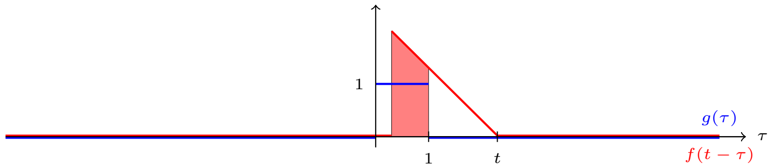
$$(f * g)(t) = \frac{t^2}{2} \qquad \text{(area of a triangle)}$$

• If $1 \leq t \leq 2$:



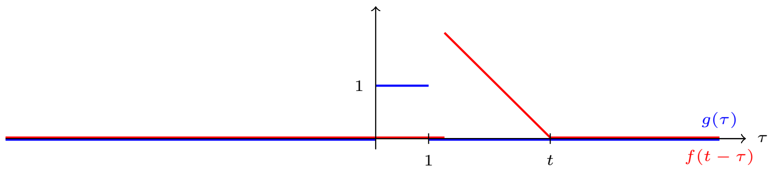
$$(f * g)(t) = \frac{t^2}{2} - \frac{(t-1)^2}{2}$$

- If $2 \leq t \leq 3$:



$$(f * g)(t) = \frac{2^2}{2} - \frac{(t-1)^2}{2}$$

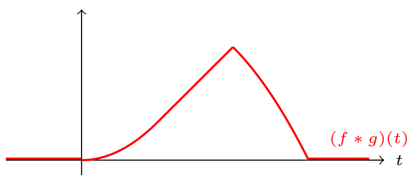
- If $3 \leq t$:



$$(f * g)(t) = 0$$

Thus we get

$$(f * g)(t) = \begin{cases} 0 & \text{, for } t \leq 0 \\ \frac{x^2}{2} & \text{, for } 0 < t \leq 1 \\ \frac{x^2}{2} - \frac{(x-1)^2}{2} & \text{, for } 1 < t \leq 2 \\ 2 - \frac{(x-1)^2}{2} & \text{, for } 2 < t \leq 3 \\ 0 & \text{, for } 3 < t. \end{cases}$$



9.4. Filters

Filter

Definition 317.
A *filter* is a

- continuous,
- linear and

- time invariant system.

Filters and transfer functions

Let us consider a filter $A : X \to Y$ (in the analog case). If we use as an input signal a harmonic function

$$e_\lambda(t) := e^{2\pi i \lambda t}, \qquad (t \in \mathbb{R}),$$

the corresponding output signal will be written as $f_\lambda(t)$; so by definition

$$f_\lambda(t) := A(e_\lambda(t)), \qquad (t \in \mathbb{R}).$$

Remark that $e_\lambda(t+\tau) \equiv e_\lambda(t) \cdot e_\lambda(\tau)$. Hence, by time invariance we get

$$f_\lambda(t+\tau) = A(e_\lambda(t+\tau)) = A(e_\lambda(t) \cdot e_\lambda(\tau)).$$

By linearity, this leads to

$$f_\lambda(t+\tau) = e_\lambda(t) \cdot A(e_\lambda(\tau)) = e_\lambda(t) \cdot f_\lambda(\tau), \qquad \forall \tau \in \mathbb{R}.$$

Setting $\tau = 0$ we get

$$f_\lambda(t) = f_\lambda(0) \cdot e_\lambda(t), \qquad \forall t \in \mathbb{R}.$$

Definition 318.
The function

$$H : \mathbb{R} \to \mathbb{C}, \lambda \mapsto f_\lambda(0)$$

is called the *transfer function of the system.*

Proposition 319.
For any LTI-system we have

$$A(e_\lambda(t)) = H(\lambda) \cdot e_\lambda(t), \qquad (t \in \mathbb{R}),$$

where

$$e_\lambda(t) := e^{2\pi i \lambda t},$$

i.e. $e_\lambda(t)$ *is an eigenvector of* A *with eigenvalue* $H(\lambda)$.

Consider now a T-periodic input signal; such signals can be written, as we will show it and under suitable hypotheses, as

$$\sum_{k=-\infty}^{+\infty} c_k e_{\frac{k}{T}}(t) = \sum_{k=-\infty}^{+\infty} c_k e^{2\pi i k \frac{t}{T}}.$$

The corresponding output signal will be T-periodic, too and, due to the continuity of the filter, we get the following output signal:

$$\sum_{k=-\infty}^{+\infty} H\left(\frac{k}{T}\right) c_k e_{\frac{k}{T}}(t) = \sum_{k=-\infty}^{+\infty} H\left(\frac{k}{T}\right) c_k e^{2\pi i k \frac{t}{T}}$$

Thus, for T-periodic signals, the filter reduces to a 'multiplication by $H\left(\frac{k}{T}\right)$ operator' acting on the so called Fourier coefficients c_k.

If the input signal is not periodic, it can be written, under suitable hypotheses, as

$$\int_{-\infty}^{+\infty} \hat{f}(\lambda) e^{2\pi i \lambda t}\, d\lambda.$$

The corresponding output signal will be (by the continuity of the filter):

$$\int_{-\infty}^{+\infty} H(\lambda) \cdot \hat{f}(\lambda) e^{2\pi i \lambda t}\, d\lambda.$$

Thus, the filter reduces to a 'multiplication by $H(\lambda)$ operator' acting on the so called Fourier transformed $\hat{f}$.

The tool of central interest

We see, that the tool of central interest is *Fourier's representation of signals*

9.5. Fourier's representation of periodic signals

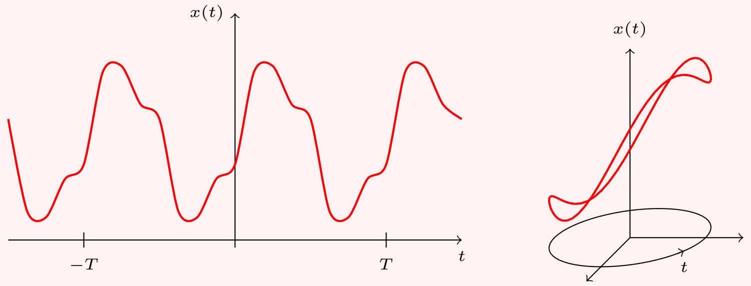
Definition 320.

Given: An analog signal x

we say: this signal is T-*periodic* (with $T > 0$ a fixed constant) iff:

$$x(t+T) = x(t), \qquad \forall t \in \mathbb{R}.$$

We visualize T-periodic signals frequently as a function defined on the circle $\mathbb{T}_T$ with circumference T.



Remark that the harmonic signal

$$t \mapsto e^{2\pi i \lambda t}$$

is T periodic if and only if

$$e^{2\pi i \lambda (t+T)} = e^{2\pi i \lambda T} \cdot e^{2\pi i \lambda t} \equiv e^{2\pi i \lambda t}$$

i.e. if and only if

$$e^{2\pi i \lambda T} = 1$$

i.e. if and only if

$$\lambda = \frac{k}{T} \qquad \text{, for some } k \in \mathbb{Z}.$$

Thus we get

Proposition 321.

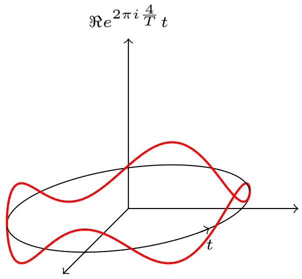
The signals

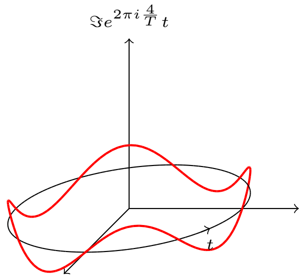
$$t \mapsto e^{2\pi i \frac{k}{T} t} = \cos\left(2\pi i \frac{k}{T} t\right) + i \cdot \sin\left(2\pi i \frac{k}{T} t\right)$$

with $k \in \mathbb{Z}$ are all T-periodic.

Conversely, any harmonic signal $e^{2\pi i \lambda t}$ is T-periodic if and only if it is of the above form for some $k \in \mathbb{Z}$.

For $k = 4$ the real and imaginary part of the T-periodic harmonics are given by





9.5.1. The Fourier coefficients

Fourier (like Euler, Lagrange and D. Bernoulli before him) discovered that T*-periodic and suitably regular* function can be synthesized as

$$f(t) = \sum_{-\infty}^{+\infty} c_k \cdot e^{2\pi i \frac{k}{T} t}.$$

This equation is called the synthesis equation.

Herein, complex Fourier coefficients c_k can be obtained, under suitable hypotheses, by the analysis equation

$$c_k = \frac{1}{T} \int_0^T f(t) \cdot e^{-2\pi i \frac{k}{T} t}\, dt = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \cdot e^{-2\pi i \frac{k}{T} t}\, dt$$

An alternative representation of T-periodic signals is obtained, by splitting the above representation in the real and the imaginary part:

$$f(t) = \frac{a_0}{2} + \sum_{1}^{+\infty} \left[a_k \cdot cos\left(2\pi i \frac{k}{T} t\right) + b_k \cdot sin\left(2\pi i \frac{k}{T} t\right) \right],$$

where

$$\begin{aligned} a_0 &= 2 \cdot c_0 = \frac{2}{T} \int_0^T f(t)\, dt \\ a_k &= c_k + c_{-k} = \frac{2}{T} \int_0^T f(t) \cdot \cos\left(2\pi i \frac{k}{T} t\right)\, dt \\ b_k &= i(c_k - c_{-k}) = \frac{2}{T} \int_0^T f(t) \cdot \sin\left(2\pi i \frac{k}{T} t\right)\, dt \end{aligned}$$

Remark that the c_k can be expressed by a_k and b_k in the following way

$$c_k = \begin{cases} \frac{a_{-k} + i \cdot b_{-k}}{2} & \text{, if } k < 0 \\ \frac{a_0}{2} & \text{, if } k = 0 \\ \frac{a_k - i \cdot b_k}{2} & \text{, if } k > 0. \end{cases}$$

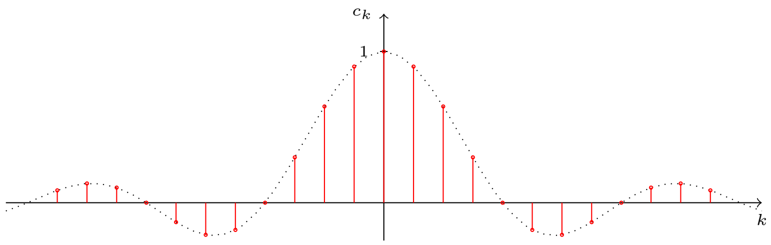
Example 322.
For the T-periodic signal given by

$$f(t) = \begin{cases} 4 & , \text{if } t \in [-\frac{T}{8}, \frac{T}{8}] \\ 0 & , \text{if } t \in [-\frac{T}{2}, -\frac{T}{8}[\cup]\frac{T}{8}, \frac{T}{2}] \end{cases}.$$

one has

$$c_k = \frac{4}{k\pi} \sin(k\pi/4) \qquad , \text{for } k \neq 0$$

and $c_0 = 1$.



Our aim is now to prove the synthesis equation

$$f(t) = \sum_{-\infty}^{+\infty} c_k \cdot e^{2\pi i \frac{k}{T} t}.$$

for a T-periodic signal, where the Fourier coefficients are given by

$$c_k = \frac{1}{T} \int_0^T f(t) \cdot e^{-2\pi i \frac{k}{T} t}\, dt = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \cdot e^{-2\pi i \frac{k}{T} t}\, dt$$

9.5.2. Dirac sequences

Definition 323.

Given: a sequence $\{\Delta_n\}_{n=1}^{+\infty}$ of functions

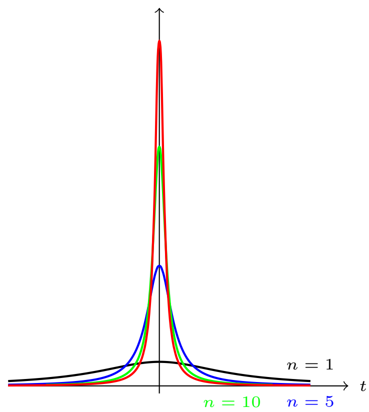
$$\Delta_n : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \Delta_n(t)$$

we say: this sequence is a *Dirac sequence* iff:

1. $\Delta_n(t) \geq 0, \forall t \in \mathbb{R} \quad (n \in \{1, 2, 3, \ldots\})$;
2. Every function Δ_n is absolutely integrable (in the sense of Lebesgue, see below) and
$$\frac{1}{T} \cdot \int_{-\infty}^{+\infty} \Delta_n(t)\, dt = 1.$$
3. $\forall \delta > 0$ kept fixed,
$$\int_{|t| \geq \delta} \Delta_n(t)\, dt \to 0 \quad \text{as } n \to \infty.$$

Example 324.

$$\Delta_n(t) = \frac{n}{\pi} \cdot \frac{1}{1 + n^2 t^2}.$$



Proposition 325.

<u>*Hyp*</u> *Let*

$$f : \mathbb{R} \to \mathbb{C}, t \mapsto f(t)$$

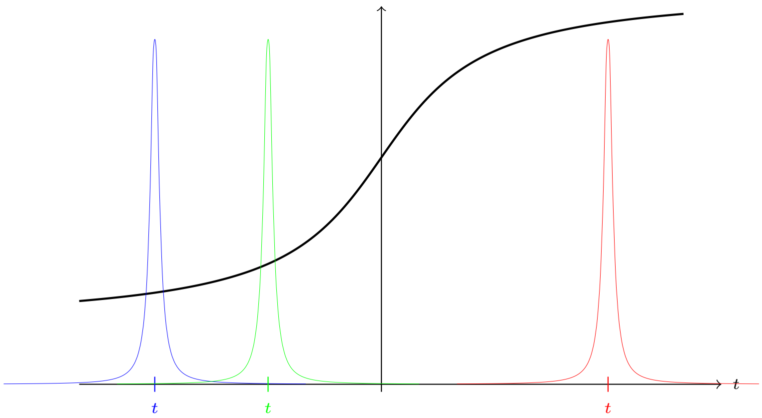
be a continuous and bounded *function.*

<u>*Concl*</u> *For any Dirac sequence* $\{\Delta_n\}_{n=1}^{+\infty}$*, we have*

$$\lim_{n\to\infty} (\Delta_n * f)(t) = f(t), \qquad \forall t \in \mathbb{R}.$$

Moreover, this convergence is uniform on any bounded and closed subset K of $\mathbb{R}$:

$$\sup_{t\in K} |(\Delta_n * f)(t) - f(t)| \to 0 \qquad , \textit{as } n \to \infty.$$



Definition 326.

Given: a sequence $\{\Delta_n\}_{n=1}^{+\infty}$ of functions

$$\Delta_n : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \Delta_n(t)$$

we say: this sequence is a *T-periodic Dirac sequence* iff:

1. every function $\Delta_n(\cdot)$ is T-periodic;
2. $\Delta_n(t) \geq 0$, $\forall t \in \mathbb{R} \quad (n \in \{1, 2, 3, \ldots\})$;
3. Every function Δ_n is integrable (in the sense of Lebesgue, see below) over one period and

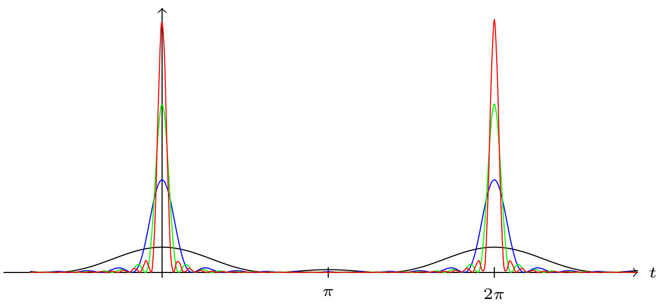
$$\int_{-T/2}^{T/2} \Delta_n(t)\, dt = 1.$$

4. $\forall \delta > 0$ kept fixed, $\int_{\delta \leq |t| \leq T/2} \Delta_n(t)\, dt \to 0$ as $n \to \infty$.

Example 327.

$$\Delta_n(t) = \frac{\sin^2(n \cdot t/2)}{2\pi \cdot n \cdot \sin^2(t/2)}$$

defines a 2π-periodic Dirac sequence:



Proposition 328.

<u>*Hyp*</u> *Let*

$$f : \mathbb{R} \to \mathbb{C}, t \mapsto f(t)$$

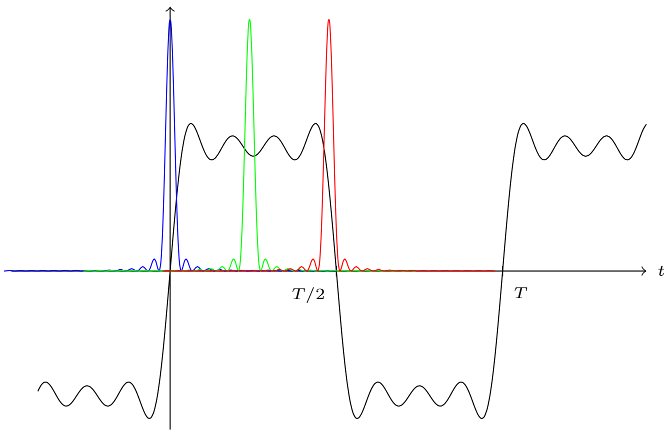
be a continuous, T-periodic *function.*

<u>*Concl*</u> *Then, for any T-periodic Dirac sequence $\{\Delta_n\}_{n=1}^{+\infty}$, we have*

$$\lim_{n\to\infty} (\Delta_n * f)(t) = f(t), \qquad \forall t \in \mathbb{R}.$$

Moreover, this convergence is uniform on $\mathbb{R}$:

$$\sup_{t\in\mathbb{R}} |(\Delta_n * f)(t) - f(t)| \to 0 \qquad \text{, as } n \to \infty.$$



9.5.3. Dirichlet kernel

For a T-periodic (analog) signal, we consider the partial sum

$$S_N(f)(t) := \sum_{k=-N}^{N} c_k \cdot e^{2\pi i \frac{k}{T} t},$$

for $N \in \{1, 2, 3, 4, \ldots\}$, where the c_k are the Fourier coefficents:

$$c_k = \frac{1}{T} \int_0^T f(\tau) \cdot e^{-2\pi i \frac{k}{T}\tau} \, d\tau.$$

Putting all together, we get

$$\begin{aligned} S_N(f)(t) &:= \sum_{k=-N}^{N} \left[\frac{1}{T} \int_0^T f(\tau) \cdot e^{-2\pi i \frac{k}{T}\tau} \, d\tau \right] e^{2\pi i \frac{k}{T} t} \\ &= \frac{1}{T} \int_0^T f(\tau) \cdot \sum_{k=-N}^{N} e^{2\pi i \frac{k}{T}(t-\tau)} \, d\tau \end{aligned}$$

We consider the sequence $\{D_N\}_{N=1}^{+\infty}$ of so called *Dirichlet kernels* with

$$D_n(t) := \frac{1}{T} \cdot \sum_{k=-N}^{N} e^{2\pi i \frac{k}{T} t}.$$

Then

$$\boxed{S_N(f)(t) = (f * D_N)(t).}$$

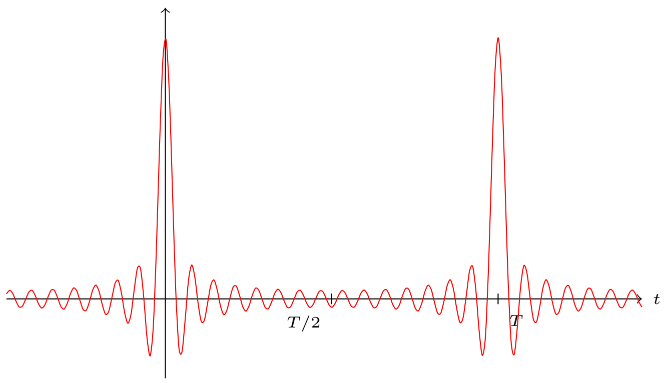
We have

$$\begin{aligned} T \cdot D_N(t) &= \sum_{k=-N}^{N} e^{2\pi i \frac{k}{T} t} = e^{-2\pi i \frac{N}{T} t} \cdot \frac{1 - e^{2\pi i \frac{2N+1}{T} t}}{1 - e^{2\pi i \frac{1}{T} t}} \\ &= \frac{e^{2\pi i \frac{N+1/2}{T} t} - e^{-2\pi i \frac{N+1/2}{T} t}}{e^{2\pi i \frac{1/2}{T} t} - e^{-2\pi i \frac{1/2}{T} t}}, \end{aligned}$$

i.e.

$$\boxed{D_n(t) = \begin{cases} \frac{2N+1}{T} & \text{, if } t \in \{k \cdot T \mid k \in \mathbb{Z}\} \\ \frac{1}{T} \cdot \frac{\sin(\frac{(2N+1)\cdot \pi}{T} t)}{\sin(\frac{\pi}{T} t)} & \text{, elsewhere.} \end{cases}}$$

Unfortunately, the sequence of Dirichlet kernels is not a Dirac sequence, since these kernels are not non-negative:



9.5.4. Fejér kernels

Let us put

$$(T_N f)(t) := \frac{1}{N} \sum_{k=1}^{N} (S_k f)(t) = \frac{(S_1 f)(t) + (S_2 f)(t) + \cdots + (S_N f)(t)}{N}.$$

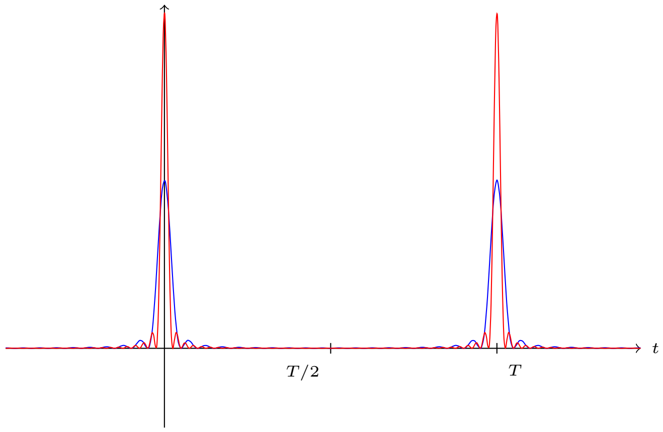
A computation similar to the above one leads to

$$\boxed{(T_N f)(t) = (f * F_N)(t), \qquad \text{for all } t \in \mathbb{R}}$$

where

$$F_N(t) = \begin{cases} \frac{1}{T} \cdot \frac{\sin^2(\frac{N\pi}{T}t)}{N \cdot \sin^2(\frac{\pi}{T}t)} & \text{, if } t \in \{k \cdot T \mid k \in \mathbb{Z}\} \\ \frac{N}{T} & \text{, elsewhere.} \end{cases}$$

This time, the sequence $\{F_N\}_{N=1}^{+\infty}$ of Fejér kernels is a Dirac sequence:



In fact, one can verify that

$$\int_0^T F_N(t)\, dt = 1.$$

Proposition 329.

Hyp *Let* $f : \mathbb{R} \to \mathbb{C}$ *be a* continuous, T-periodic *signal.*

Concl *We have*

$$\lim_{N \to +\infty} (f * F_N)(t) = f(t), \qquad \forall t \in \mathbb{R}.$$

Moreover, the above convergence is uniform*:*

$$\lim_{N \to \infty} \sup_{t \in \mathbb{R}} |(f * F_N)(t) - f(t)| = 0.$$

9.5.5. A counter example

As we have seen above, the sequence of Dirichlet kernels $\{D_N\}_{N=1}^{+\infty}$ is not a Dirac sequence. Thus, we cannot conclude that

$$\sum_{k=-\infty}^{+\infty} c_k \cdot e^{2\pi i \frac{k}{T} t} \equiv f(t)$$

for all continuous, T-periodic functions f, if we choose c_k as $c_k = \frac{1}{T}\int_0^T f(t)\cdot e^{-2\pi i \frac{k}{T}t}\,dt$.

We can only replace the sequence

$$s_n := \sum_{k=-n}^{n} c_k \cdot e^{2\pi i \frac{k}{T}t}$$

by an averaged sequence

$$\sigma_n := \frac{s_1 + s_2 + \cdot + s_n}{n}$$

and then $f(t) = \lim_{n\to\infty}\sigma_n$

Even worse, there exist continuous, T-periodic signals whose corresponding Fourier series does not converge to f:

Proposition 330.
There exist continuous, T-periodic signals whose corresponding Fourier series does not converge everywhere to f.

9.5.6. Positive results

The lack of positivity of the Dirichlet kernels can be counter-balanced by more smoothness of the signals.

More precisely, we have the following two positive results:

Proposition 331.
If the signal $f:\mathbb{R}\to\mathbb{C}$ is T-periodic and of class C^1, *then*

$$\lim_{N\to\infty}(S_N f)(t) = f(t), \qquad \forall t\in\mathbb{R}.$$

Moreover, this convergence is uniform:

$$\lim_{N\to\infty}\sup_{t\in\mathbb{R}}|(f * D_N)(t) - f(t)| = 0.$$

Proposition 332.
Let $f:\mathbb{R}\to\mathbb{C}$ be a piecewise C^1 signal, such that at the points of discontinuities, the unilateral limits of the derivatives exist (Dini's condition).

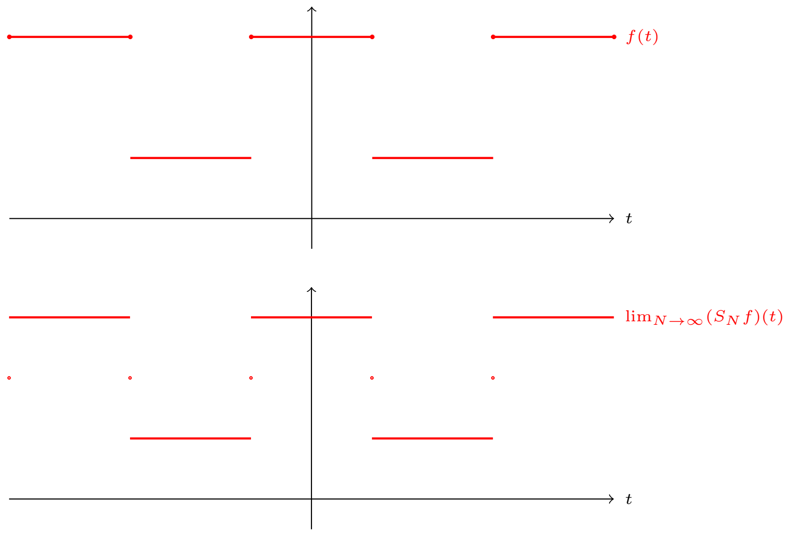
Then

$$\lim_{N\to\infty}(S_N f)(t) = \frac{f(t^-)+f(t^+)}{2}, \qquad \forall t\in\mathbb{R}.$$

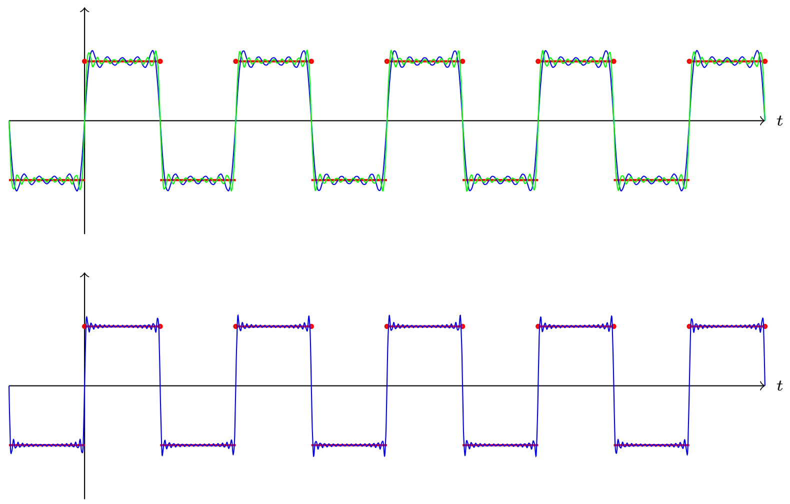
Thus, at points of continuity, we have

$$\lim_{N\to\infty}(S_N f)(t) = f(t),$$

whereas at jump points, the limit gives the mean value at the jump.



Gibb's phenomenon at jump points



Part III

Spaces with inner product

10

Pre-Hilbert and Hilbert spaces

10.1. Pre-Hilbert spaces

10.1.1. Inner product

Definition 333.
Given: a linear space X over $\mathbb{K}$
we define: an *inner product* on X as:
a mapping

$$\langle \cdot \mid \cdot \rangle : X \times X \to \mathbb{K}, \quad (u, v) \mapsto \langle u \mid v \rangle$$

having the following properties:

1. **Strict positivity:** One has

$$\langle u \mid u \rangle \geq 0, \qquad \forall u \in X$$

and

$$\langle u \mid u \rangle = 0 \iff u = 0.$$

We write

$$\boxed{\|u\| := \sqrt{\langle u \mid u \rangle}, \qquad \text{for } u \in X.}$$

and we will eventually show that this is a norm on X.

2. **Linearity in the first slot:** We have

$$\langle \alpha u + v \mid w \rangle = \alpha \langle u \mid w \rangle + \langle v \mid w \rangle, \qquad \forall \alpha \in \mathbb{K}, \quad \forall u, v, w \in X.$$

3. **Symmetry:** We have

$$\langle u \mid v \rangle = \overline{\langle v \mid u \rangle}, \qquad \forall u, v \in X.$$

Remark that, if $\mathbb{K} = \mathbb{R}$, this reduces to

$$\langle u \mid v \rangle = \langle v \mid u \rangle, \qquad \forall u, v \in X.$$

Definition 334.
A linear space X equipped with an inner product $\langle \cdot \mid \cdot \rangle$ is called a *pre-Hilbert space*.

Proposition 335.

<u>*Hyp*</u> *Let $(X, \langle \cdot \mid \cdot \rangle)$ be a pre-Hilbert space over $\mathbb{K}$.*
<u>*Concl*</u> *The inner product is anti-linear in the second slot:*

$$\langle u \mid \alpha v + w \rangle = \overline{\alpha} \langle u \mid v \rangle + \langle u \mid w \rangle , \qquad \forall \alpha \in \mathbb{K}, \quad \forall u, v, w \in X.$$

If $\mathbb{K} = \mathbb{R}$, this reduces to linearity in the second slot:

$$\langle u \mid \alpha v + w \rangle = \alpha \langle u \mid v \rangle + \langle u \mid w \rangle , \qquad \forall \alpha \in \mathbb{K}, \quad \forall u, v, w \in X.$$

Proof. By symmetry and linearity in the first slot, we have

$$\begin{aligned} \langle u \mid \alpha v + w \rangle &= \overline{\langle \alpha v + w \mid u \rangle} \\ &= \overline{\alpha \langle v \mid u \rangle + \langle w \mid u \rangle} \\ &= \overline{\alpha} \cdot \overline{\langle v \mid u \rangle} + \overline{\langle w \mid u \rangle} \\ &= \overline{\alpha} \cdot \langle u \mid v \rangle + \langle u \mid w \rangle . \end{aligned}$$

□

Example 336.
$\mathbb{R}^p$ is a pre-Hilbert space if one sets

$$\langle x \mid y \rangle := \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_p \end{bmatrix} \cdot \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_p \end{bmatrix} = \sum_{k=1}^{p} \xi_k \eta_k = \xi_1 \cdot \eta_1 + \cdots + \xi_p \cdot \eta_p.$$

Example 337.
$\mathbb{C}^p$ is a pre-Hilbert space if one sets

$$\langle x \mid y \rangle := \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_p \end{bmatrix} \cdot \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_p \end{bmatrix} = \sum_{k=1}^{p} \xi_k \overline{\eta_k} = \xi_1 \cdot \overline{\eta_1} + \cdots + \xi_p \cdot \overline{\eta_p}$$

Proposition 338.
In every pre-Hilbert space $(X, \langle \cdot \mid \cdot \rangle$ *one has*

$$\langle u \mid 0 \rangle = \langle 0 \mid u \rangle = 0, \qquad \forall u \in X.$$

Proof. This follows from

$$\langle u \mid 0 \rangle = \langle u \mid u - u \rangle = \langle u \mid u \rangle - \langle u \mid u \rangle = 0.$$

□

10.1.2. Schwarz inequality

Schwarz inequality

Proposition 339.
In every pre-Hilbert space $(X, \langle \cdot \mid \cdot \rangle)$*, one has*

$$|\langle u \mid v \rangle| \leq \sqrt{\langle u \mid u \rangle} \cdot \sqrt{\langle u \mid u \rangle}, \qquad \forall u, v \in X$$

i.e.

$$|\langle u \mid v \rangle| \leq \|u\| \cdot \|v\|, \qquad \forall u, v \in X.$$

Proof. **(I) Case where $v = 0$:**
One has

$$|\langle u \mid 0 \rangle| = 0 = \sqrt{\langle u \mid u \rangle} \cdot \underbrace{\sqrt{\langle 0 \mid 0 \rangle}}_{=0}.$$

So, the Schwarz inequality is in fact an equality.
Remark that the same argument can be used for the case where $u = 0$.
(II): Case where $v \neq 0$ and $\mathbb{K} = \mathbb{R}$:
We have

$$\begin{aligned} 0 &\leq \langle u - \alpha v \mid u - \alpha v \rangle \\ &= \langle u \mid u \rangle - 2\alpha \langle u \mid v \rangle + \alpha^2 \langle v \mid v \rangle =: f(\alpha). \end{aligned}$$

Let us choose α in such a way that the function f achieves its minimum; let us choose

$$\alpha = \frac{\langle u \mid v \rangle}{\langle v \mid v \rangle}.$$

Then we get

$$0 \leq \langle u \mid u \rangle - 2 \cdot \frac{\langle u \mid v \rangle}{\langle v \mid v \rangle} \cdot \langle u \mid v \rangle + \frac{\langle u \mid v \rangle^2}{\langle v \mid v \rangle^2} \langle v \mid v \rangle$$

i.e.

$$0 \leq \frac{\langle u \mid u \rangle \cdot \langle v \mid v \rangle - \langle u \mid v \rangle^2}{\langle v \mid v \rangle}$$

Since $\langle v \mid v \rangle > 0$, this implies

$$\langle u \mid v \rangle^2 \leq \langle u \mid u \rangle \cdot \langle v \mid v \rangle$$

and

$$|\, \langle u \mid v \rangle \,| \leq \sqrt{\langle u \mid u \rangle} \cdot \sqrt{\langle v \mid v \rangle}.$$

(III): Case where $v \neq 0$ and $\mathbb{K} = \mathbb{C}$:
We have

$$\begin{aligned} 0 &\leq \langle u - \alpha v \mid u - \alpha v \rangle \\ &= \langle u \mid u \rangle - \alpha \langle v \mid u \rangle - \overline{\alpha} \langle u \mid v \rangle + \alpha \cdot \overline{\alpha} \cdot \langle v \mid v \rangle =: f(\alpha). \end{aligned}$$

Let us choose α as above

$$\alpha = \frac{\langle u \mid v \rangle}{\langle v \mid v \rangle}.$$

Then we get

$$\begin{aligned} 0 &\leq \langle u \mid u \rangle - 2\frac{\langle u \mid v \rangle \cdot \overline{\langle u \mid v \rangle}}{\langle v \mid v \rangle} \langle v \mid u \rangle + \frac{\langle u \mid v \rangle \cdot \overline{\langle u \mid v \rangle}}{\langle v \mid v \rangle} \\ 0 &\leq \langle u \mid u \rangle - \frac{|\langle u \mid v \rangle|^2}{\langle v \mid v \rangle} \end{aligned}$$

Thus, since $\langle v \mid v \rangle \geq 0$, we get the claim

$$|\langle u \mid v \rangle|^2 \leq \langle u \mid u \rangle \cdot \langle v \mid v \rangle\,.$$

□

10.1.3. Orthogonality

By the Schwarz inequality

$$|\, \langle u \mid v \rangle \,| \leq \sqrt{\langle u \mid u \rangle} \cdot \sqrt{\langle u \mid u \rangle} = \|u\| \cdot \|v\|$$

one has, for pre-Hilbert spaces over $\mathbb{R}$,

$$-1 \leq \frac{\langle u \mid v \rangle}{\|u\| \cdot \|v\|}.$$

Thus, one may define the angle α between u and v through

$$\cos \alpha = \frac{\langle u \mid v \rangle}{\|u\| \cdot \|v\|}$$

and

$$0 \leq \alpha \leq \pi.$$

Such consideration can help as a motivation for the following definition.

Definition 340.
Two elements u and v in a pre-Hilbert space $(X, \langle \cdot \mid \cdot \rangle)$ are called *orthogonal* if

$$\langle u \mid v \rangle = 0.$$

If moreover

$$\|u\| = \|v\| = 1, \qquad i.e. \qquad \langle u \mid u \rangle = \langle v \mid v \rangle = 1,$$

these orthogonal elements are called *orthonormed*.

10.1.4. Norm generated by an inner product

We have yet introduced in the pre-Hilbert space$X$, as a notation,

$$\|u\| := \sqrt{\langle u \mid u \rangle}, \qquad \forall u \in X.$$

It turns out, that this defines a norm on X: thus, every pre-Hilbert space is a normed space:

Proposition 341.

Hyp *Let $(X, \langle \cdot \mid \cdot \rangle)$ be a pre-Hilbert space.*
Concl *Then*

$$\|u\| := \sqrt{\langle u \mid u \rangle}, \qquad \forall u \in X$$

defines a norm on X, so that X can be considered as a normed space, too .

Proof. We must check that the above defined $\|\cdot\|$ has all properties of a norm:

(I) Strict positivity:
We have, for all $u \in H$,

$$\langle u \mid u \rangle \geq 0 \Longrightarrow \|u\| \geq 0.$$

Moreover,

$$\|u\| = 0 \Longleftrightarrow \langle u \mid u \rangle = 0 \Longleftrightarrow u = 0.$$

(II) Homogeneity:
A direct computations shows that $\|\cdot\|$ is homogeneous. Indeed, $\forall \alpha \in \mathbb{K}$ and $\forall u \in X$, we have

$$\begin{aligned} \|\alpha u\| &= \sqrt{\langle \alpha u \mid \alpha u \rangle} = \sqrt{\alpha \cdot \overline{\alpha} \cdot \langle u \mid u \rangle} \\ &= \sqrt{\alpha \cdot \overline{\alpha} \cdot \|u\|^2} = \sqrt{|\alpha|^2} \cdot \sqrt{\|u\|^2} \\ &= |\alpha| \cdot \|u\|. \end{aligned}$$

(III) Triangular inequality:
For all u and $v \in X$, we have by Schwarz inequality

$$\begin{aligned}\|u+v\|^2 &= \langle u+v \mid u+v\rangle \\ &= \langle u \mid u\rangle + \underbrace{\langle u \mid v\rangle + \overline{\langle u \mid v\rangle}}_{=2\Re\langle u \mid v\rangle} + \langle v \mid v\rangle \\ &\leq \|u\|^2 + 2\cdot\|u\|\cdot\|v\| + \|v\|^2 \\ &= (\|u\|+\|v\|)^2,\end{aligned}$$

and this gives

$$\|u+v\| \leq \|u\| + \|v\|.$$

□

Remark 342. *Thus, all properties of a normed space remain valid in a pre-Hilbert space.*
In particular, we may speak of convergence. We say that a sequence $\{u_n\}_{n=1}^{+\infty}$ *in a pre-Hilbert space* $(X, \langle\cdot \mid \cdot\rangle)$ *converges to some* $u \in X$ *if*

$$\lim_{n\to\infty} \|u_n - u\| = 0,$$

i.e. if

$$\lim_{n\to\infty} \sqrt{\langle u_n - u \mid u_n - u\rangle} = 0.$$

Moreover, we can discuss notions like continuity. As an example, we show that the inner product is continuous.

Continuity of the inner product

Proposition 343.
In any pre-Hilbert space $(X, \langle\cdot \mid \cdot\rangle)$*, the inner product*

$$\langle\cdot \mid \cdot\rangle : X \times X \to \mathbb{K}$$

is continuous with respect to the induced norm

$$\|\cdot\| := \sqrt{\langle\cdot \mid \cdot\rangle}.$$

Thus, if $\{u_n\}_{n=1}^{+\infty}$ *and* $\{v_n\}_{n=1}^{+\infty}$ *are two convergent sequences in the pre-Hilbert space* X*, then*

$$\left.\begin{array}{l}\lim_{n\to\infty} u_n = u \\ \lim_{n\to\infty} v_n = v\end{array}\right\} \Longrightarrow \lim_{n\to\infty} \langle u_n \mid v_n\rangle = \langle u \mid v\rangle.$$

Proof. The claim follows from

$$\begin{aligned}
|\langle u_n \mid v_n\rangle - \langle u \mid v\rangle| &= |\langle u_n - u \mid v_n\rangle + \langle u \mid v_n - v\rangle| \\
&\leq |\langle u_n - u \mid v_n\rangle| + |\langle u \mid v_n - v\rangle| \\
&\leq \underbrace{\|u_n - u\|}_{\to 0} \cdot \underbrace{\|v_n\|}_{\text{bounded}} + \|u\| \cdot \underbrace{\|v_n - v\|}_{\to 0} \\
\lim_{n\to\infty} |\langle u_n \mid v_n\rangle - \langle u \mid v\rangle| &= 0.
\end{aligned}$$

□

Definition 344.

Given: a subset $M \subset X$ of a normed space X (for example a pre-Hilbert space)

we say: *M is dense in X* iff:
every point u in X can be approximated to any precision by points in M, i.e. M is dense in X iff

$\forall u \in X$
$\exists$ a convergent sequence $\{u_n\}_{n=1}^{+\infty}$ in M with
$\lim_{n\to\infty} u_n = u$.

Proposition 345.

Hyp *Let $M \subset X$ be a dense set in the pre-Hilbert space $(X, \langle\cdot \mid \cdot\rangle)$. (Thereby we explicitly do* not *exclude the case where $M = X$.)*

Concl *If $u \in X$ is such that*

$$\langle u \mid v\rangle = 0, \quad \forall v \in M,$$

then $v = 0$.

Proof. There exists a sequence $\{u_n\}_{n=1}^{+\infty}$ in M with

$$\lim_{n\to\infty} u_n = u.$$

Thus

$$\langle u \mid u_n\rangle = 0, \qquad \forall n$$

and

$$\|u\|^2 = \langle u \mid u\rangle = \lim_{n\to\infty} \langle u \mid u_n\rangle = 0,$$

i.e. $u = 0$. □

10.1.5. Polarization

As we have seen it, every pre-Hilbert space $(X, \langle\cdot \mid \cdot\rangle)$ can be equipped with a norm $\|\cdot\|$ generated by the inner product trough

$$\|u\| := \sqrt{\langle u \mid u\rangle}, \qquad u \in X.$$

The somewhat surprising fact is now, that conversely, the inner product can be expressed by this norm.

Proposition 346.

<u>*Hyp*</u> *Consider a pre-Hilbert space $(X, \langle\cdot \mid \cdot\rangle)$ over $\mathbb{K}$ equipped with the norm*

$$\|u\| := \sqrt{\langle u \mid u\rangle}, \qquad u \in X$$

generated by the inner product.

<u>*Concl*</u> *Then the inner product can be expressed by the norm:*

1. *If $\mathbb{K} = \mathbb{R}$, we have*

$$\langle u \mid v\rangle = \frac{1}{4} \cdot \left[\|u+v\|^2 - \|u-v\|^2\right], \qquad \forall u, v \in X.$$

2. *If $\mathbb{K} = \mathbb{C}$, we have*

$$\begin{aligned}\langle u \mid v\rangle \; = \; & \frac{1}{4} \cdot \Big[\|u+v\|^2 - \|u-v\|^2 \\ & \quad + i\left(\|u+iv\|^2 - \|u-iv\|^2\right)\Big], \qquad \forall u, v \in X.\end{aligned}$$

Proof. This follows from

$$\begin{array}{ll} \|u+v\|^2 & = \|u\|^2 + \|v\|^2 + 2\Re\langle u \mid v\rangle \\ \|u-v\|^2 & = \|u\|^2 + \|v\|^2 - 2\Re\langle u \mid v\rangle \\ \hline \|u+v\|^2 - \|u-v\|^2 & = 4\Re\langle u \mid v\rangle \end{array}$$

and from

$$\begin{array}{ll} \|u+iv\|^2 & = \|u\|^2 + \|v\|^2 + 2\Im\langle u \mid v\rangle \\ \|u-iv\|^2 & = \|u\|^2 + \|v\|^2 - 2\Im\langle u \mid v\rangle \\ \hline \|u+iv\|^2 - \|u-iv\|^2 & = 4\Im\langle u \mid v\rangle \end{array}$$

□

10.1.6. The parallelogram rule

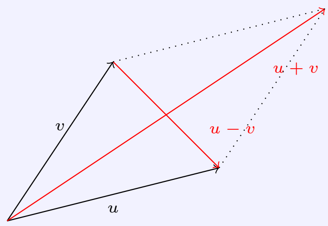
The above result calls for a simple question: Can every normed space $(X, \|\cdot\|)$ be transformed into a pre-Hilbert space equipped with an inner product defined with the help of the identities in the above proposition.

Before giving an answer, we establish a nice (and important) property of pre-Hilbert spaces.

Parallelogram rule

Proposition 347.
In any pre-Hilbert space $(X, \langle\cdot \mid \cdot\rangle)$, the following parallelogram rule holds:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2, \qquad \forall u, v \in X.$$



Proof. This follows from

$$\begin{array}{ll} \|u+v\|^2 & = \langle u+v \mid u+v\rangle = \|u\|^2 + \|v\|^2 + 2\Re\langle u \mid v\rangle \\ \|u-v\|^2 & = \langle u-v \mid u-v\rangle = \|u\|^2 + \|v\|^2 - 2\Re\langle u \mid v\rangle \\ \hline \|u+v\|^2 + \|u-v\|^2 & = 2\|u\|^2 + 2\|v\|^2. \end{array}$$

□

The following example shows that the parallelogram rule does not hold in all normed spaces. Thus, not all normed spaces can be transformed into pre-Hilbert spaces.

Example 348.
We consider in the Banach space $L^1([0,1], \mathscr{B}(\mathbb{R})|_{[0,1]}, \lambda^1|_{[0,1]})$ two functions

$$u(x) = 1 \qquad \text{and} \qquad v(x) = \frac{1}{2} - x.$$

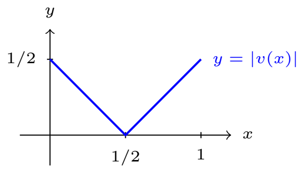
Then

- We have

$$2\|u\|_{L^1}^2 = 2\left[\text{(R)-}\int_0^1 1\,dx\right]^2 = 2$$

and

$$2\|v\|_{L^1}^2 = 2\left[\text{(R)-}\int_0^1 \left|\frac{1}{2} - x\right|\,dx\right]^2 = \frac{1}{8}$$



So

$$2\|u\|_{L^1}^2 + 2\|v\|_{L^1}^2 = \frac{17}{8}.$$

- On the other hand, we have

$$\|u+v\|_{L^1}^2 = \left[\int_0^1 \underbrace{\left| \frac{3}{2} - x \right|}_{=\frac{3}{2}-x} dx \right]^2 = 1$$

and

$$\|u-v\|_{L^1}^2 = \left[\int_0^1 \underbrace{\left| \frac{1}{2} + x \right|}_{=\frac{1}{2}+x} dx \right]^2 = 1.$$

So

$$\|u+v\|_{L^1}^2 + \|u-v\|_{L^1}^2 = 2.$$

Hence

$$\|u+v\|_{L^1}^2 + \|u-v\|_{L^1}^2 \neq 2\|u\|_{L^1}^2 + 2\|v\|_{L^1}^2.$$

This means that there is no inner product on $L^1([0,1], \mathscr{B}(\mathbb{R})|_{[0,1]}, \lambda^1|_{[0,1]})$ that generates the norm $\|\cdot\|_{L^1}$.

Or formulated in a different way, the product one could define through the polarization identities does not define an inner product.

So a central question arises:
When is a given normed space $(X, \|\cdot\|)$ in fact a pre-hilbert space $(X, \langle \cdot \mid \cdot \rangle)$ with

$$\|u\| = \sqrt{\langle u \mid u \rangle}, \qquad \forall u \in X?$$

The following proposition gives an answer!

Proposition 349.

Hyp *Suppose that $(X, \|\cdot\|)$ is a normed space over $\mathbb{K}$, whose norm satisfies the parallelogram rule:*

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2, \qquad \forall u, v \in X.$$

- *If $\mathbb{K} = \mathbb{R}$ we put, for u and $v \in X$,*

$$\langle u \mid v \rangle := \frac{1}{4} \cdot \left[\|u+v\|^2 + \|u-v\|^2\right].$$

- *If $\mathbb{K} = \mathbb{C}$ we put, for u and $v \in X$,*

$$\langle u \mid v \rangle := \frac{1}{4} \cdot \left[\|u+v\|^2 + \|u-v\|^2 + i\left(\|u+iv\|^2 + \|u-iv\|^2\right)\right].$$

Concl *Then,*

1. *$\langle \cdot \mid \cdot \rangle$ is an inner product on X;*
2. *$(X, \langle \cdot \mid \cdot \rangle)$ is thus a pre-Hilbert space.*

10.2. Hilbert spaces

10.2.1. Complete pre-Hilbert spaces

Definition 350.
A *Hilbert space (over $\mathbb{K}$)* is a pre-Hilbert space $(X, \langle \cdot \mid \cdot \rangle)$ (over $\mathbb{K}$) that is complete with respect to the induced norm

$$\|u\| = \sqrt{\langle u \mid u \rangle}, \qquad \forall u \in X.$$

Remark 351. *Thus, every Hilbert space is a Banach space. So all that was said about Banach spaces remains valid in Hilbert spaces.*

Consider a pre-Hilbert space:

$(X, \langle\cdot \mid \cdot\rangle)$ a pre-Hilbert space

$\|u\| := \sqrt{\langle u \mid u\rangle}, \quad \forall u \in X$ $\downarrow$

$(X, \|\cdot\|)$ is a normed space where the parallelogram rule holds

completion with respect to $\|\cdot\|$ $\downarrow$

$(\tilde{X}, \|\cdot\|)$ is a Banach space with $X \subset \tilde{X}$ and where $\|\cdot\|$ has be extended by continuity.
Thus the parallelogram rule continues to hold.

define an inner product by polarization with this extended norm $\downarrow$

$(\tilde{X}, \langle\cdot \mid \cdot\rangle)$ is a Hilbert space with $X \subset \tilde{X}$ and where $\langle\cdot \mid \cdot\rangle$ has be extended by continuity

Proposition 352.

Hyp *Let $X, \langle\cdot \mid \cdot\rangle)$ be a pre-Hilbert space with associated norm*

$$\|u\| := \sqrt{\langle u \mid u\rangle}, \qquad \forall \in X.$$

Concl *The completion of $(X, \|\cdot\|)$ is a Hilbert space $(\tilde{X}, \langle\cdot \mid \cdot\rangle)$ where the inner product is extended by continuity.*

10.2.2. Examples of Hilbert spaces

Example 353.
If we equip $\mathbb{C}^N$, for $N = 1, 2, 3, \ldots$, with the "usual" inner product

$$\langle x \mid y\rangle := \sum_{k=1}^{N} \xi_k \cdot \overline{\eta_k},$$

where $x = (\xi_i, \ldots, \xi_N)$ and $y = (\eta_1, \ldots, \eta_N)$, we get a Hilbert space, since $\mathbb{C}^N$ equipped with the norm

$$\|x\| := \sqrt{\langle x \mid x\rangle} = \sqrt{\sum_{k=1}^{N} |\xi_k|^2}$$

is a Banach space.

Example 354.

If we equip $\mathbb{R}^N$, for $N = 1, 2, 3, \ldots$, with the "usual" inner product

$$\langle x \mid y\rangle := \sum_{k=1}^{N} \xi_k \cdot \eta_k,$$

where $x = (\xi_i, \ldots, \xi_N)$ and $y = (\eta_1, \ldots, \eta_N)$, we get a Hilbert space, since $\mathbb{R}^N$ equipped with the norm

$$\|x\| := \sqrt{\langle x \mid x\rangle} = \sqrt{\sum_{k=1}^{N} \xi_k^2}$$

is a Banach space.

Example 355.

Let us consider, for $-\infty < a < b < +\infty$, the space

$$C[a, b] := \{u : [a, b] \to \mathbb{R} \ : \ u \text{ is continuous}\}$$

and put, for $u(\cdot)$ and $v(\cdot) \in C[a, b]$,

$$\langle u \mid v\rangle := \int_a^b u(x)v(x)\, dx.$$

(I) $(C[a, b], \langle\cdot \mid \cdot\rangle)$ **is a pre-Hilbert space.**
Indeed, $\langle\cdot \mid \cdot\rangle$ is an inner product, since

- We have
$$\langle u \mid u\rangle = \int_a^b u(x)^2\, dx \geq 0, \qquad \forall u(\cdot) \in C[a, b]$$
and
$$\langle u \mid u\rangle = 0 \iff u = 0 \text{ a.e.} \iff u(x) = 0 \text{ for } x \in [a, b].$$

- For $u(\cdot)$, $v(\cdot)$ and $w(\cdot) \in C[a, b]$ and for $\alpha \in \mathbb{R}$, we have
$$\begin{aligned} \langle \alpha \cdot u + v \mid w\rangle &= \int_a^b (\alpha \cdot u(x) + v(x)) \cdot w(x)\, dx \\ &= \alpha \int_a^b u(x) \cdot w(x)\, dx + \int_a^b v(x) \cdot w(x)\, dx \\ &= \alpha \cdot \langle u \mid w\rangle + \langle v \mid w\rangle\,. \end{aligned}$$

- Moreover, for all $u(\cdot)$ and $v(\cdot) \in C[a, b]$, we have
$$\langle u \mid v\rangle = \int_a^b u(x) \cdot v(x)\, dx = \int_a^b v(x) \cdot u(x)\, dx = \langle v \mid u\rangle\,.$$

(II) However, $(C[a,b], \langle \cdot \mid \cdot \rangle)$ is not a Hilbert space since it is not complete.
Remark that the norm generated by the inner product

$$\|u\|_{L^2} = \sqrt{\langle u \mid u \rangle} = \sqrt{\int_a^b u(x)^2 \, dx}$$

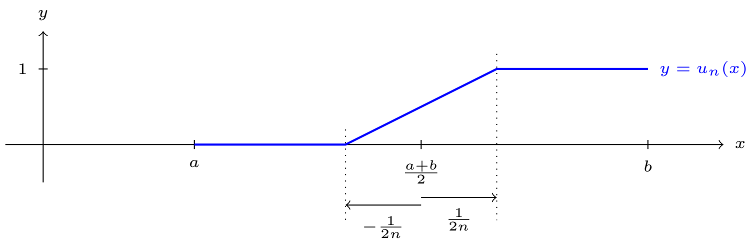
is the standard norm on $L^2([a,b], \mathscr{L}(\mathbb{R})|_{[a,b]}, \lambda^1|_{[a,b]})$ and that $C[a,b] \subset L^2([a,b], \mathscr{L}(\mathbb{R})|_{[a,b]}, \lambda^1|_{[a,b]})$.

We give now a sequence $\{u_n\}_{n=n_0}^{+\infty}$ in $C[a,b]$ that converges in $L^2([a,b], \mathscr{L}(\mathbb{R})|_{[a,b]}, \lambda^1|_{[a,b]})$ to a limit function u. Thus this sequence is a Cauchy sequence. Then we show that the limit function u does not belong to $C[a,b]$. Thus the pre-Hilbert space $(C[a.b], \langle \cdot \mid \cdot \rangle)$ is not complete and not a Hilbert space.

The above announced sequence $\{u_n\}_{n=n_0}^{+\infty}$ is given by

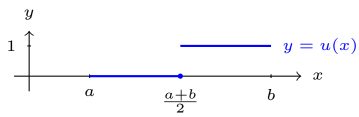
$$u_n(x) = \begin{cases} 0 & \text{, for } a \leq x \leq \frac{a+b}{2} - \frac{1}{2n} \\ nx & \text{, for } \frac{a+b}{2} - \frac{1}{2n} < x \frac{a+b}{2} + \frac{1}{2n} \\ 1 & \text{, for } \frac{a+b}{2} + \frac{1}{2n} \leq x \leq b. \end{cases}$$

Thereby we assume that n is large enough, say $n > \frac{1}{b-a}$.



This sequence converges to the limit function

$$u(x) = \begin{cases} 0 & \text{, for } a \leq x \leq \frac{a+b}{2} \\ 1 & \text{, for } \frac{a+b}{2} < x \leq b. \end{cases}$$

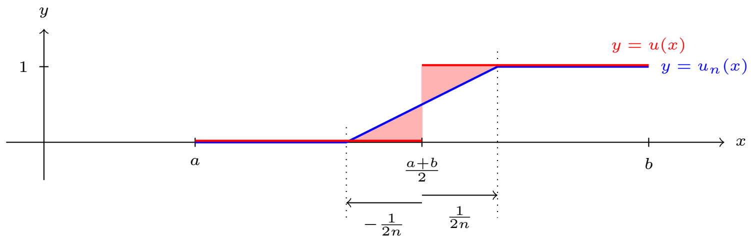


Indeed

$$\|u_n - u\|_{L^2} = \int_a^b \left(u_n(x) - u(x)\right)^2 \, dx \leq \int_{\frac{a+b}{2} - \frac{1}{2n}}^{\frac{a+b}{2} + \frac{1}{2n}} 1 \, dx \leq \frac{1}{n}$$

so that

$$\lim_{n \to \infty} \|u_n - u\|_{L^2} = 0.$$



However,

$$u \notin C[a, b].$$

Thus the Cauchy sequence does not converge in $(C[a, b], \langle \cdot \mid \cdot \rangle)$.

Let us remark that the completion of $(C[a, b], \langle \cdot \mid \cdot \rangle)$ is the Hilbert space $L^2([a, b], \mathscr{L}(\mathbb{R})|_{[a,b]}, \lambda^1|_{[a,b]})$

10.2.3. The Hilbert spaces $L^2(X, \mathscr{A}, \mu)$ and $L^2_{\mathbb{C}}(X, \mathscr{A}, \mu)$

Consider the Banach space

$$L^2(X, \mathscr{A}, \mu) \quad \text{with the norm } \|u\|_{L^2} := \left[\int_X |u(x)|^2 \, d\mu(x) \right]^{1/2}$$

If one puts

$$\langle u \mid v \rangle_{L^2} := \int_X u(x) v(x) \, d\mu(x)$$

one gets an inner product that generates the above norm $\| \cdot \|_{L^2}$. Remark thereby that for example

$$\langle u \mid u \rangle_{L^2} = 0 \Longleftrightarrow u = 0 \ \mu\text{-a.e.} \Longleftrightarrow u = 0.$$

Since $L^2(X, \mathscr{A}, \mu)$ is complete with respect to the generated norm $\| \cdot \|_{L^2}$, $L^2(X, \mathscr{A}, \mu)$ is a Hilbert space:

Proposition 356.

$L^2(X, \mathscr{A}, \mu)$ equipped with the inner product

$$\langle u \mid v \rangle_{L^2} := \int_X u(x) \cdot v(x) \, d\mu(x)$$

is a Hilbert space over $\mathbb{R}$.

Example 357.

As typical examples of such Hilbert spaces, let us mention:

- $L^2(\mathbb{R}) := L^2(\mathbb{R}, \mathscr{L}(\mathbb{R}), \lambda^1)$;

- $L^2([0,T]) := L^2([0,T], \mathscr{L}(\mathbb{R})|_{[0,T]}, \lambda^1|_{0,T})$ that models real-valued T-periodic signals;
- $\ell^2(\mathbb{Z}) := \left\{\text{(double sided) sequence } \{f_n\}_{n\in\mathbb{Z}} \text{ in } \mathbb{R} \;:\; \sum_{k\in\mathbb{Z}} f_n^2 < +\infty\right\}$, where
  - $\displaystyle\sum_{k\in\mathbb{Z}} f_n^2 = \lim_{N,M\to+\infty} \sum_{k=-M}^{N} f_n^2$
  - $\displaystyle\sum_{k\in\mathbb{Z}} f_n^2 = \int_{\mathbb{Z}} f_n^2 \, d\mu(n) \quad (\text{with } \mu(A) = |A|);$
  - $\displaystyle\langle \{f_n\} \mid \{g_n\}\rangle_{\ell^2} := \sum_{k\in\mathbb{Z}} f_n \cdot g_n;$
  - $\displaystyle\|\{f_n\}\|_{\ell^2} = \sum_{k\in\mathbb{Z}} f_n^2.$

Consider the Banach space

$$L^2_{\mathbb{C}}(X, \mathscr{A}, \mu) \quad \text{with the norm } \|u\|_{L^2} := \left[\int_X |u(x)|^2 \, d\mu(x)\right]^{1/2}$$

If one puts

$$\langle u \mid v\rangle_{L^2} := \int_X u(x)\overline{v(x)} \, d\mu(x)$$

one gets an inner product that generates the above norm $\|\cdot\|_{L^2}$. Remark thereby that for example

$$\langle u \mid u\rangle_{L^2} = 0 \iff u = 0 \; \mu\text{-a.e.} \iff u = 0.$$

Since $L^2_{\mathbb{C}}(X, \mathscr{A}, \mu)$ is complete with respect to the generated norm $\|\cdot\|_{L^2}$, $L^2_{\mathbb{C}}(X, \mathscr{A}, \mu)$ is a Hilbert space:

Proposition 358.
$L^2_{\mathbb{C}}(X, \mathscr{A}, \mu)$ equipped with the inner product

$$\langle u \mid v\rangle_{L^2} := \int_X u(x) \cdot \overline{v(x)} \, d\mu(x)$$

is a Hilbert space over $\mathbb{C}$.

Remark that this inner product has the required symmetry property:

$$\begin{aligned}\langle u \mid v\rangle &= \int_X u(x) \cdot \overline{v(x)} \, d\mu(x) = \int_X \overline{\overline{u(x)} \cdot v(x)} \, d\mu(x) \\ &= \overline{\int_X \overline{u(x)} \cdot v(x) \, d\mu(x)} = \overline{\langle v \mid u\rangle}.\end{aligned}$$

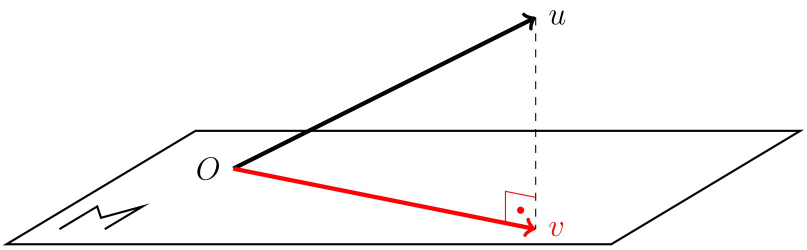
Example 359.
As typical examples of such Hilbert spaces, let us mention:

- $L^2_{\mathbb{C}}(\mathbb{R}) := L^2_{\mathbb{C}}(\mathbb{R}, \mathscr{L}(\mathbb{R}), \lambda^1)$;
- $L^2_{\mathbb{C}}([0,T]) := L^2([0,T], \mathscr{L}(\mathbb{R})|_{[0,T]}, \lambda^1|_{0,T})$ that models complex-valued T-periodic signals;
- $\ell^2_{\mathbb{C}}(\mathbb{Z}) := \big\{(\text{double sided) sequence } \{f_n\}_{n\in\mathbb{Z}} \text{ in } \mathbb{C} \ : \ \sum_{k\in\mathbb{Z}} |f_n|^2 < +\infty\big\}$, where
  - $\displaystyle\sum_{k\in\mathbb{Z}} |f_n|^2 = \lim_{N,M\to+\infty} \sum_{k=-M}^{N} |f_n|^2$
  - $\displaystyle\sum_{k\in\mathbb{Z}} |f_n|^2 = \int_{\mathbb{Z}} |f_n|^2 \, d\mu(n) \quad (\text{with } \mu(A) = |A|)$;
  - $\displaystyle\langle \{f_n\} \mid \{g_n\}\rangle_{\ell^2} := \sum_{k\in\mathbb{Z}} f_n \cdot \overline{g_n}$;
  - $\displaystyle\|\{f_n\}\|_{\ell^2} = \sum_{k\in\mathbb{Z}} |f_n|^2.$

10.2.4. Orthogonal projections on Hilbert spaces

We consider the following minimizing problem

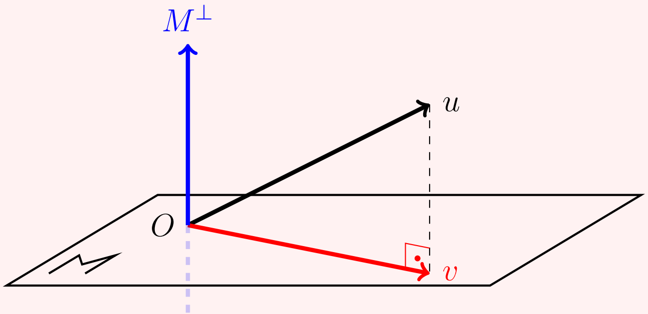
Find $v \in M$ such that
$$\|u - v\| = \inf\{\|u - w\| \ : \ w \in M\}$$
under the following assumptions:
- M is a *closed* linear subspace of the Hilbert space $(X, \langle\cdot \mid \cdot\rangle)$
- $u \in X$ is kept fixed.



Remark 360. *Let us remark that* M*, as a* closed *subspace of the Hilbert space* $(X, \langle\cdot \mid \cdot\rangle)$*, is itself a Hilbert space (with respect to the same inner product* $\langle\cdot \mid \cdot\rangle$

Definition 361.
We denote by $M^{\perp}$ the orthogonal complement of M:

$$M^{\perp} := \{w \in X \ : \ \langle w \mid v \rangle = 0 \text{ for all } v \in M\}.$$



Proposition 362.

Hyp: *Suppose that M is a* closed *linear subspace of the Hilbert space $(X, \langle \cdot \mid \cdot \rangle)$ over $\mathbb{K}$.*
Let $u \in X$ be a given (and fixed) element.

Concl:

1. *The problem*

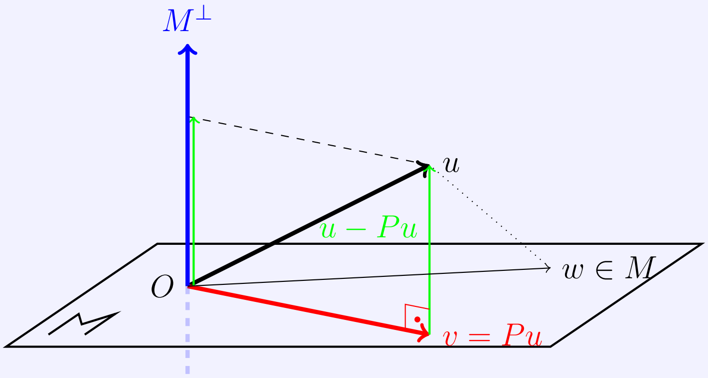
Find $v \in M$ such that $\|u - v\| = \inf\{\|u - w\| \ : \ w \in M\}$

   has a unique solution

   $$v := Pu \in M$$

   and

   $$u - v \in M^{\perp}$$

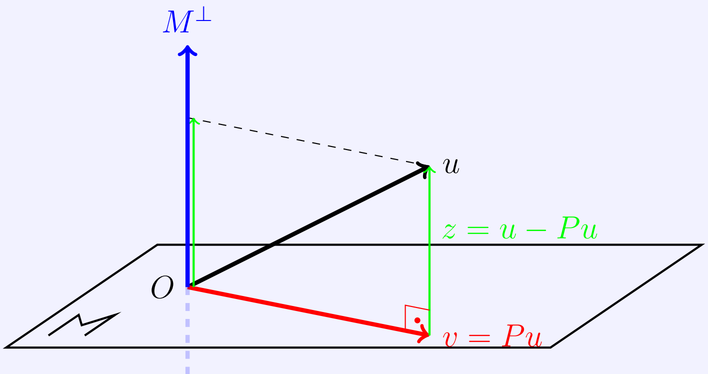


2. *There is a unique decomposition of u of the form*

$$u = v + z \qquad \text{,where } v \in M \text{ and } z \in M^\perp.$$

with $v = Pu$ and $z = u - Pu$. We write

$$X = M \oplus M^\perp.$$



The proof will use the parallelogram rule: thus the above minimizing problem cannot be solved in a Banach space that is not Hilbert. In fact, in Banach spaces, the above minimizing problem can only be solved if for example $\dim M < \infty$; and in Banach spaces, we cannot speak about the orthogonal complement or the orthogonal decomposition.

Proof. **(I) An equivalent formulation for our minimizing problem:**
Since, for $v \in M$, we have

$$\begin{aligned} \|u - v\|^2 &= \langle u - v \mid u - v \rangle \\ &= \|u\|^2 - 2\Re \langle u \mid v \rangle + \|v\|^2, \end{aligned}$$

we can formulate our minimizing problem as follows:

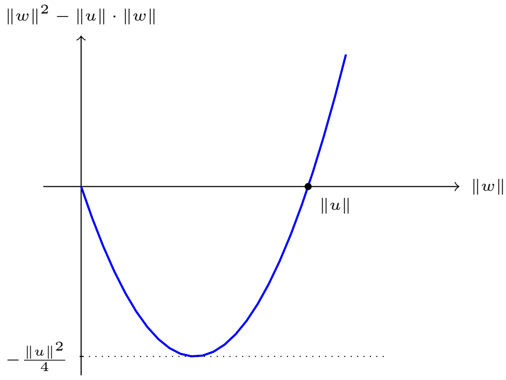
> For the fixed element $u \in X$,
> find $v \in M$ such that
> $$G(v) = \inf_{w \in M} G(w),$$
> where $G(w) := \|w\|^2 - 2\Re \langle u \mid w \rangle$

Since

$$G(w) \geq \|w\|^2 - \|u\| \cdot \|w\| = \|w\| \cdot [\|w\| - \|u\|],$$

We get

$$\inf_{w \in M} G(w) =: \alpha > -\infty.$$



(II): A minimizing sequence:

Thus, we may choose an so-called minimizing sequence $\{w_n\}_{n=1}^{+\infty}$ in M. By that we mean a sequence of elements $w_n \in M$ with

$$G(w_n) \searrow \alpha, \qquad \text{as } n \to \infty.$$

Remark that this minimizing sequence $\{w_n\}_{n=1}^{+\infty}$ is a Cauchy sequence.

Indeed, by the parallelogram rule, we have

$$\begin{array}{ll}
G(w_n) & = \|w_n\|^2 - 2\Re \langle u \mid w_n \rangle \\
G(w_m) & = \|w_m\|^2 - 2\Re \langle u \mid w_m \rangle \\
\hline
2G(w_n) + 2G(w_m) & = \|w_n + w_m\|^2 + \|w_n - w_m\|^2 - 4\Re \langle u \mid w_n + w_m \rangle \\
4G\left(\frac{w_n + w_m}{2}\right) & = \|w_n + w_m\|^2 \qquad\qquad\qquad\quad - 4\Re \langle u \mid w_n + w_m \rangle \\
\hline
\end{array}$$

i.e.

$$2G(w_n) + 2G(w_m) - 4G\left(\frac{w_n + w_m}{2}\right) = \|w_n - w_m\|^2$$

But

$$\begin{aligned}
G(w_n) &= \alpha + o(1), && \text{as } n \to \infty \\
G(w_m) &= \alpha + o(1), && \text{as } m \to \infty \\
G\left(\frac{w_n + w_m}{2}\right) &\geq \alpha.
\end{aligned}$$

Thus given any $\varepsilon > 0$, we have for n and m large enough (say $n, m \geq n_0 = n_0(\varepsilon)$),

$$\begin{aligned} 0 \leq \|w_n - w_m\|^2 &= 2G(w_n) + 2G(w_m) - 4G\left(\frac{w_n + w_m}{2}\right) \\ &\leq 4\alpha + \varepsilon - 4\alpha = \varepsilon. \end{aligned}$$

Thus the minimizing sequence $\{w_n\}_{n=1}^{+\infty}$ is a Cauchy sequence.

(III) The limit of the minimizing sequence is a minimizer:
Thus, the minimizing sequence is converging, say

$$\lim_{n\to\infty} w_n := v \in M.$$

By continuity of the functional G, we have

$$G(v) = G(\lim_{n\to\infty} w_n) = \lim_{n\to\infty} G(w_n) = \alpha.$$

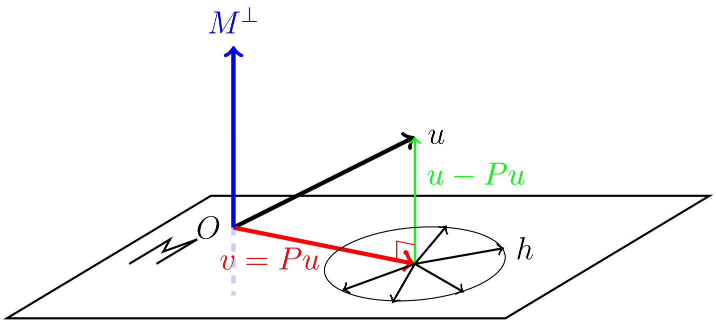
Thus, the limit of the minimizing sequence is a minimizer of G.

This means that $v = Pu$ is the element that achieves

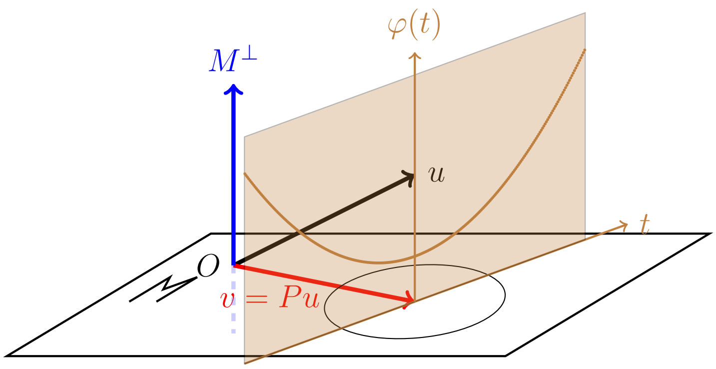
$$\inf_{w\in M} \|u - w\|.$$

(IV) So it remains to show that $u - v \in M^{\perp}$.
To show this, we consider, for each fixed $h \in M$ the functional

$$\begin{aligned} \varphi(t) &= \|u - (v + t \cdot h)\|^2, \qquad \text{where } v = Pu \\ &= \|(u - v) + t \cdot h\|^2 \\ &= \|u - v\|^2 + t \cdot 2\Re \langle u - v \mid h\rangle + t^2 \cdot \|h\|^2 \end{aligned}$$



Since v is a minimizer, $\varphi(t)$ is minimal for $t = 0$, this being so for any choice of $h \in M$.



Thus
$$0 = \frac{d}{dt}\varphi(t)|_{t=0} = 2\Re \langle u - v \mid h\rangle = 0, \qquad \forall h \in M.$$
If $\mathbb{K} = \mathbb{R}$, we get
$$\langle u - v \mid h\rangle = 0, \qquad \forall h \in M,$$
so $u - v \in M^\perp$.

If $\mathbb{K} = \mathbb{C}$, we may consider ih instead of h, and we get
$$\Re \langle u - v \mid ih\rangle = \Im \langle u - v \mid h\rangle = 0, \qquad \forall h \in M.$$
Putting all together, we get again
$$\langle u - v \mid h\rangle = 0, \qquad \forall h \in M,$$
so $u - v \in M^\perp$.

This closes the proof! □

10.2.5. Linear functionals and Riesz Theorem

We consider, for a given Hilbert space $(X, \langle \cdot \mid \cdot\rangle)$ over $\mathbb{K}$ the corresponding dual space that we denote by X^*:
$$X^* := \{f : X \to \mathbb{K} \ : \ f \text{ is bounded an linear}\}.$$
Recall that we have introduced the notation
$$f(u) =: \langle f, u\rangle.$$
When equipped with the norm
$$\|f\| := \sup_{\|u\|=1} \langle f, u\rangle,$$
the space $(X^*, \|\cdot\|)$ is a Banach space.

Our aim is to identify the dual space X^* and to show that this dual space is Hilbert, too.

Example 363.

Consider a Hilber space $(X, \langle \cdot \mid \cdot\rangle)$ over $\mathbb{K}$.

Then, any given $v \in X$ can be considered as a bounded, linear functional f_v through
$$\langle f_v, u\rangle := \langle u \mid v\rangle, \qquad \forall u \in X,$$
since $\langle \cdot \mid v\rangle$ is linear in the first slot and since
$$|\langle f_v, u\rangle| = |\langle u \mid v\rangle| \le \underbrace{\|v\|}_{=\|f_v\|} \cdot \|u\|.$$

Thus, identifying f_v and v, we may write

$$X \subset X^*.$$

Surprisingly, the following proposition shows that the inverse inclusion holds, too.

Riesz representation theorem

Proposition 364.

Hyp Consider a Hilbert space $(X, \langle\cdot \mid \cdot\rangle$ and its dual space $(x^*, \|\cdot\|)$ (as a Banach space).

Concl Then,

$$\forall f \in X^*$$
$$\exists! v \in X \text{ such that}$$
$$\langle f, u\rangle = \langle u \mid v\rangle, \quad \forall u \in X.$$

Moreover, $\|f\| = \|v\|$, where

$$\|f\| = \|f\|_{X^*} \qquad \text{and} \qquad \|v\| = \|v\|_X.$$

Remark

can now

Proof. **(I) The element $v \in X$ is uniquely determined by the bounded, linear functional f:**

Indeed, suppose on the contrary that two elements v_1 and $v_2 \in X$ exist for the same $f \in X^*$. Then

$$\langle u \mid v_1\rangle = \langle u \mid v_2\rangle, \qquad \forall u \in X$$

imply

$$\langle u \mid v_1 - v_2\rangle = 0, \qquad \forall u \in X.$$

Choosing in this relation $u = v_1 - v_2$, we get

$$\|v_1 - v_2\|^2 = \langle v_1 - v_2 \mid v_1 - v_2\rangle = 0,$$

so that $v_1 = v_2$. **(II) What remains to be shown:**

The above example has shown that any $v \in X$ can be identified with an element $f_v \in X^*$. Thus it remains to show that

$$\forall f \in X^*, \quad \exists v \in X \text{ with } \langle f, \cdot\rangle = \langle\cdot \mid v\rangle.$$

Remark that we can take $v = 0$ if $f = 0$. Thus it remains to prove the above claim for $f \neq 0$. **(II) $\ker(f)$ is a closed, linear subspace of X:**

Indeed, let $\{u_n\}_{n=1}^{+\infty}$ be a converging sequence in

$$\ker(f) := \left\{u \in X \;:\; \langle f, u\rangle = 0\right\} = f^{-1}(0)\right\}.$$

Then, since f is continuous, we have

$$\langle f, u\rangle = \langle f, \lim_{n\to\infty} u_n\rangle = \lim_{n\to\infty} \underbrace{\langle f, u_n\rangle}_{=0} = 0,$$

so that $u \in \ker(f)$. Thus $\ker(f)$ is closed.

The fact that $\ker(f)$ is a linear subspace is standard, since f is a linear mapping. **(III) If $f \neq 0$, there is an element $u_0 \in \ker(f)^\perp \setminus \{0\}$:**

If $f \neq 0$, there exists an element $u_0 \in \ker(f)^\perp \setminus \{0\}$, for else

$$\ker(f)^\perp = \{0\} \qquad \text{and thus} \qquad \ker(f) = X$$

so that

$$\langle f, u\rangle = 0, \qquad \forall u \in X$$

i.e. $f = 0$.

Thus

$$\langle f, u_0\rangle \neq 0$$

and

$$\left\langle f, \frac{1}{\langle f, u_0\rangle} \cdot u_0 \right\rangle = 1$$

(III) Decomposition of X with respect to $\ker(f)$ and $\ker(f)^\perp$:

We get in this way the following relation

$$\left\langle f, \frac{\langle f, u\rangle}{\langle f, u_0\rangle} \cdot u_0 \right\rangle = \langle f, u\rangle \qquad \forall u \in X$$

that we write as

$$\left\langle f, u - \frac{\langle f, u\rangle}{\langle f, u_0\rangle} \cdot u_0 \right\rangle = 0 \qquad \forall u \in X$$

Thus

$$u - \frac{\langle f, u\rangle}{\langle f, u_0\rangle} \cdot u_0 \in \ker(f), \qquad \forall u \in X.$$

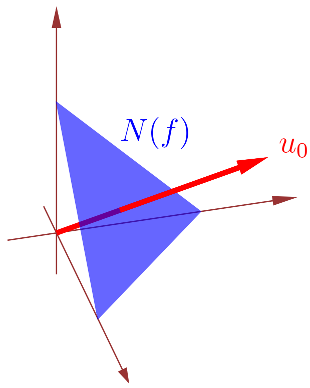
So we get the following decomposition

$$u = \underbrace{\left[u - \frac{\langle f, u\rangle}{\langle f, u_0\rangle} \cdot u_0\right]}_{\in\ker(f)} + \underbrace{\frac{\langle f, u\rangle}{\langle f, u_0\rangle} \cdot u_0}_{\in\ker(f)^\perp}, \qquad \forall u \in X.$$

Remark that in the above described decomposition

$$X = \ker(f) \oplus \underbrace{\ker(f)^\perp}_{=\text{span } u_0}$$

the dimension of $\ker(f)^\perp$ is 1.



(IV) The choice of v:
We take now

$$v = \alpha \cdot u_0, \quad \text{where } \alpha = \frac{\langle f, u_0 \rangle}{\|u_0\|^2}.$$

Then, for all $u \in X$,

$$\begin{aligned}
\langle u \mid v \rangle &= \langle u \mid \alpha u_0 \rangle = \alpha \cdot \left\langle \left[u - \frac{\langle f, u \rangle}{\langle f, u_0 \rangle} \cdot u_0 \right] + \frac{\langle f, u \rangle}{\langle f, u_0 \rangle} \cdot u_0 \mid u_0 \right\rangle \\
&= \alpha \cdot \left\langle \frac{\langle f, u \rangle}{\langle f, u_0 \rangle} \cdot u_0 \mid u_0 \right\rangle = \alpha \cdot \frac{\langle f, u \rangle}{\langle f, u_0 \rangle} \cdot \|u_0\|^2 \\
&= \langle f, u \rangle.
\end{aligned}$$

Thus we are done! □

Corollary 365.

Hyp: *Consider a Hilbert space $(X, \langle \cdot \mid \cdot \rangle$ and its dual space $(x^*, \| \cdot \|)$ (as a Banach space).*

Concl: *Then, for all $f \in X^* \setminus \{0\}$, we have*

- $\dim \ker(f)^\perp = 1$;
- *the decomposition $X = \ker(f) \oplus \ker(f)^\perp$ is given by*

$$u = \underbrace{\left[u - \frac{\langle f, u \rangle}{\langle f, u_0 \rangle} \cdot u_0 \right]}_{\in \ker(f)} + \underbrace{\frac{\langle f, u \rangle}{\langle f, u_0 \rangle} \cdot u_0}_{\in \ker(f)^\perp}, \qquad u \in X.$$

10.2.6. The duality map

Definition 366.
Given: a Hilbert space $(X, \langle \cdot \mid \cdot \rangle)$ over $\mathbb{K}$
we define: the *duality map* as:
the mapping

$$J : X \to X^*, \quad v \mapsto J(v) := \langle \cdot \mid v \rangle .$$

Remark 367. *Thus*

$$\langle J(v), u \rangle = \langle u \mid v \rangle , \qquad \forall u \in X.$$

Proposition 368.
The duality map

$$J : X \to X^*, \quad v \mapsto J(v) := \langle \cdot \mid v \rangle .$$

is

- *bijective*
- *continuous*
- *norm preserving:*

$$\|J(v)\| = \|v\|, \qquad \forall u \in X.$$

If $\mathbb{K} = \mathbb{R}$*, the duality map* J *is linear.*
If $\mathbb{K} = \mathbb{C}$*, the duality map* J *is anti-linear, i.e.*

$$J(\alpha \cdot v_1 + v_2) = \overline{\alpha} \cdot J(v_1) + J(v_2), \qquad \forall \alpha \in \mathbb{C}, \quad \forall v_1, v_2 \in X.$$

11

Bessel's inequality and equality

11.1. Orthonormal sets

Let $(X, \langle \cdot \mid \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$.

Consider a finite subset

$$\{u_1, u_2, \ldots, u_N\} \subset X$$

as well as an infinite, but countable subset

$$\{v_1, v_2, \ldots\} = \{v_n \ : \ n \in \mathbb{N}\} \subset X.$$

Definition 369.

1. $\{u_1, u_2, \ldots, u_N\} \subset X$ is a *finite, orthonormal set in* $(X, \langle \cdot \mid \cdot \rangle)$:

$$\forall k, m \in \{1, 2, \ldots, N\}, \quad \langle u_k \mid u_m \rangle = \delta_{km} = \begin{cases} 1 & \text{, if } k = m \\ 0 & \text{, if } k \neq m \end{cases}$$

2. $\{v_n \ : \ n \in \mathbb{N}\} \subset X$ is an *countably infinite, orthonormal set in* $(X, \langle \cdot \mid \cdot \rangle)$:

$$\forall k, m \in \mathbb{N}, \quad \langle u_k \mid u_m \rangle = \delta_{km} = \begin{cases} 1 & \text{, if } k = m \\ 0 & \text{, if } k \neq m \end{cases}$$

Proposition 370.

Hyp *Suppose that $\{u_1, u_2, \ldots, u_N\} \subset X$ is a finite, orthonormal set in the Hilbert space $(X, \langle \cdot \mid \cdot \rangle)$ over $\mathbb{K}$.*

<u>*Concl*</u> *If, for some $u \in X$, we have*

$$u = \sum_{k=1}^{N} \alpha_k \cdot u_k, \qquad \text{with } \alpha_k \in \mathbb{K},$$

then

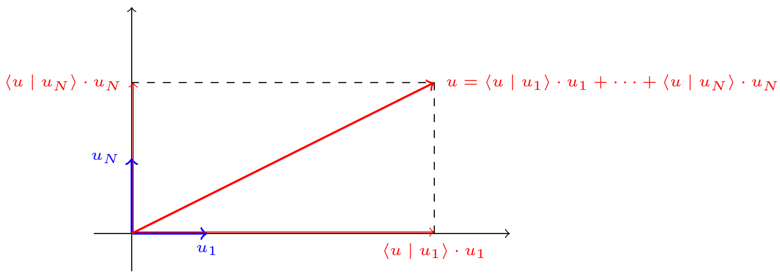
$$\alpha_k = \langle u \mid u_k \rangle, \qquad \text{for } k = 1, 2, \ldots, N,$$

i.e.

$$u = \sum_{k=1}^{N} \langle u \mid u_k \rangle \cdot u_k$$

(no other possibility!).

Proof. The proof is similar to the one given below for countably infinite, orthonormal sets. □



Proposition 371.

<u>*Hyp*</u> *Suppose that $\{u_n : n \in \mathbb{N}\} \subset X$ is a countably infinite, orthonormal set in the Hilbert space $(X, \langle \cdot \mid \cdot \rangle)$ over $\mathbb{K}$.*

<u>*Concl*</u> *If, for some $u \in X$, we have*

$$u = \sum_{k=1}^{\infty} \alpha_k \cdot u_k = \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k \cdot u_k, \qquad \text{with } \alpha_k \in \mathbb{K},$$

then

$$\alpha_k = \langle u \mid u_k \rangle, \qquad \text{for } k \in \mathbb{N},$$

i.e.

$$u = \sum_{k=1}^{\infty} \langle u \mid u_k \rangle \cdot u_k$$

(no other possibility!).

Definition 372.
We call

$$\alpha_k = \langle u \mid u_k \rangle$$

the the Fourier coefficients of u.

Proof. The claim follows from:

$$\begin{aligned}
\langle u \mid u_k \rangle &= \left\langle \lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j \cdot u_j \mid u_k \right\rangle \\
&= \lim_{n\to\infty} \left\langle \sum_{j=1}^{n} \alpha_j \cdot u_j \mid u_k \right\rangle \qquad \text{continuity of inner product} \\
&= \underbrace{\lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j \cdot \underbrace{\langle u_j \mid u_k \rangle}_{=\delta_{jk}}}_{=\alpha_k \text{ if } n \geq k} = \alpha_k.
\end{aligned}$$

□

Definition 373.
Given: a finite, orthonormal set $\{u_1, u_2, \ldots, u_N\} \subset X$ in the Hilbert space $(X, \langle \cdot \mid \cdot \rangle)$ over $\mathbb{K}$
we say: this set $\{u_1, u_2, \ldots, u_N\}$ is *complete* iff:

$$u = \sum_{k=1}^{N} \langle u \mid u_k \rangle \cdot u_k, \qquad \forall u \in X.$$

Definition 374.
Given: a countably infinite, orthonormal set $\{u_1, u_2, \ldots, \} = \{u_n : n \in \mathbb{N}\} \subset X$ in the Hilbert space $(X, \langle \cdot \mid \cdot \rangle)$ over $\mathbb{K}$
we say: this set $\{u_1, u_2, \ldots\}$ is *complete* iff:

$$u = \sum_{k=1}^{\infty} \langle u \mid u_k \rangle \cdot u_k = \lim_{n\to\infty} \sum_{k=1}^{n} \langle u \mid u_k \rangle \cdot u_k, \qquad \forall u \in X.$$

11.2. Least square method of Gauss

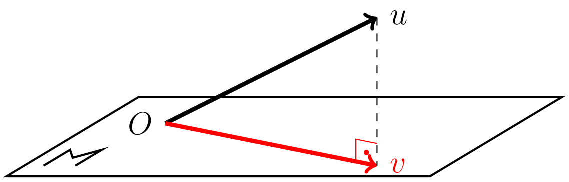
Let us consider a finite, set $\{u_1, u_2, \ldots, u_m\} \subset X$ in the Hilbert space $(X, \langle \cdot \mid \cdot \rangle)$ over $\mathbb{K}$. We put

$$M := \text{span}\,\{u_1, u_2, \ldots, u_m\}.$$

We yet know that the problem

Given any $u \in X$, find $v \in M$ such that $\quad \|u - v\| = \inf_{w \in M} \|u - w\|$

has a unique solution.



We formulate now the same problem with respect to the finite, orthonormal set $\{u_1, u_2, \ldots, u_m\}$. To this purpose, we set

$$f : \mathbb{K}^m \to \mathbb{R}, \quad f(\alpha_1, \alpha_2, \ldots, \alpha_m) := \left\| u - \sum_{k=1}^{m} \alpha_u \cdot u_k \right\|^2$$

Thus, we get the following equivalent minimization problem:

Given any $u \in X$, find $(\tilde{\alpha}_1, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) \in \mathbb{K}^m$ such that $\quad f(\tilde{\alpha}_1, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) = \inf_{(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{K}^m} f(\alpha_1, \alpha_2, \ldots, \alpha_m).$

Proposition 375.
Under the above made assumptions, there exists a unique

$$(\tilde{\alpha}_1, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) \in \mathbb{K}^m$$

such that

$$f(\tilde{\alpha}_1, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) = \inf_{(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{K}^m} f(\alpha_1, \alpha_2, \ldots, \alpha_m).$$

Moreover

$$\tilde{\alpha}_k = \langle u \mid u_k \rangle, \qquad \textit{for } k = 1, 2, \ldots, m.$$

Hence,

$$\sum_{k=1}^{m} \langle u \mid u_k \rangle \cdot u_k$$

is the best possible approximation of u inside

$$\text{span}\,\{u_1, u_2, \ldots, u_m\}$$

.

Proof. We have

$$\begin{aligned}
f(\alpha_1, \alpha_2, \ldots, \alpha_m) &= \left\langle u - \sum_{k=1}^{m} \alpha_k \cdot u_k \mid u - \sum_{j=1}^{m} \alpha_j \cdot u_j \right\rangle \\
&= \|u\|^2 - \sum_{k=1}^{m} \alpha_k \cdot \langle u_k \mid u \rangle - \\
&\quad \sum_{j=1}^{m} \overline{\alpha_j} \cdot \langle u \mid u_j \rangle + \sum_{k=1}^{m} \sum_{j=1}^{m} \alpha_k \overline{\alpha_j} \langle u_k \mid u_j \rangle \\
&= \|u\|^2 - \sum_{k=1}^{m} \alpha_k \cdot \langle u_k \mid u \rangle - \\
&\quad \sum_{k=1}^{m} \overline{\alpha_k} \cdot \langle u \mid u_k \rangle + \sum_{k=1}^{m} |\alpha_k|^2
\end{aligned}$$

Remark that

$$\begin{aligned}
|\langle u \mid u_k \rangle - \alpha_k|^2 &= \Big[\langle u \mid u_k \rangle - \alpha_k\Big] \cdot \Big[\overline{\langle u \mid u_k \rangle} - \overline{\alpha_k}\Big] \\
&= |\,\langle u \mid u_k \rangle\,|^2 - \alpha_k \cdot \overline{\langle u \mid u_k \rangle} - \overline{\alpha_k} \cdot \langle u \mid u_k \rangle + |\alpha_k|^2
\end{aligned}$$

so that

$$f(\alpha_1, \alpha_2, \ldots, \alpha_m) = \|u\|^2 - \sum_{k=1}^{m} |\,\langle u \mid u_k \rangle\,|^2 + \sum_{k=1}^{m} |\langle u \mid u_k \rangle - \alpha_k|^2$$

Thus, $f(\alpha_1, \alpha_2, \ldots, \alpha_m)$ is minimal exactly if

$$\alpha_k = \langle u \mid u_k \rangle\,, \qquad \text{for } k = 1, 2, \ldots, m.$$

□

According to Proposition 362,

$$\sum_{k=1}^{m} \langle u \mid u_k \rangle \cdot u_k$$

is the orthogonal projection Pu of u on

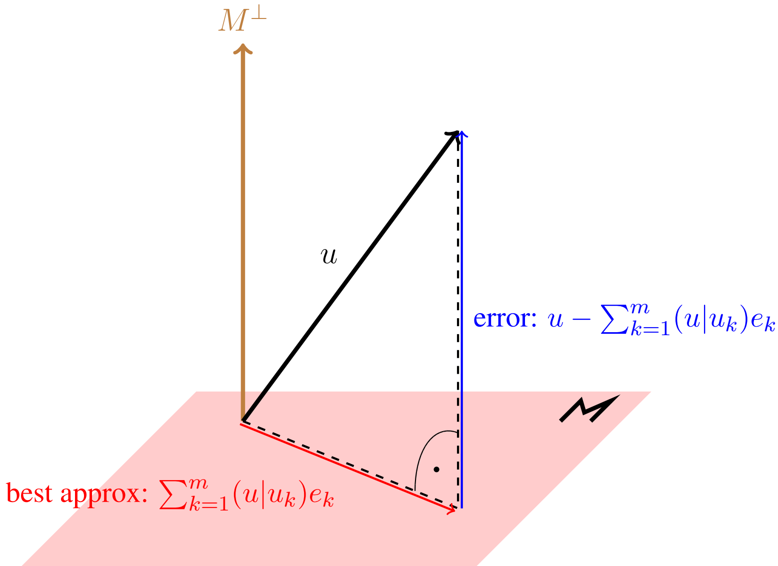
$$M := \text{span}\,\{u_1, u_2, \ldots, u_m\}.$$

The error of the approximation of u by

$$\sum_{k=1}^{m} \langle u \mid u_k \rangle \cdot u_k$$

is

$$u - \sum_{k=1}^{m} \langle u \mid u_k \rangle \cdot u_k \in M^{\perp}.$$



From the above proof, we can derive the following result:

Proposition 376.
Under the assumptions made in Proposition 375, the magnitude of the error is given by

$$\|u - \sum_{k=1}^{m} \langle u \mid u_k \rangle \cdot u_k\| = \|u\|^2 - \sum_{k=1}^{m} |\langle u \mid u_k \rangle|^2.$$

11.3. Bessel's inequality

Proposition 377.

<u>*Hyp*</u> *Consider a finite, orthonormal set* $\{u_1, u_2, \ldots, u_m\}$ *in a Hilbert space* $(X, \langle \cdot \mid \cdot \rangle)$ *over* $\mathbb{K}$.

<u>*Concl*</u> *Then*

$$\forall u \in X, \qquad \sum_{k=1}^{m} |\langle u \mid u_k\rangle|^2 \le \|u\|^2.$$

Moreover, if for some $u \in X$, we have

$$\sum_{k=1}^{m} |\langle u \mid u_k\rangle|^2 = \|u\|^2,$$

then, for this u, we have

$$u = \sum_{k=1}^{m} \langle u \mid u_k\rangle \cdot u_k.$$

Proposition 378.

<u>*Hyp*</u> *Consider a countably infinite, orthonormal set $\{u_1, u_2, \ldots\} = \{u_k : k \in \mathbb{N}\}$ in a Hilbert space $(X, \langle\cdot \mid \cdot\rangle)$ over $\mathbb{K}$.*

<u>*Concl*</u> *Then*

$$\forall u \in X$$
$$\textstyle\sum_{k=1}^{m} |\langle u \mid u_k\rangle|^2 \le \|u\|^2, \qquad \text{for } m = 1, 2, 3, \ldots$$
and
$$\textstyle\sum_{k=1}^{\infty} |\langle u \mid u_k\rangle|^2 \le \|u\|^2.$$

Moreover, if for some $u \in X$, we have

$$\sum_{k=1}^{\infty} |\langle u \mid u_k\rangle|^2 = \|u\|^2,$$

then, for this u, we have $u = \sum_{k=1}^{\infty} \langle u \mid u_k\rangle \cdot u_k$.

11.4. Bessel's equality

Corollary 379.
The countably infinite, orthonormal set $\{u_1, u_2, \ldots\} = \{u_k : k \in \mathbb{N}\}$ in a Hilbert

space $(X, \langle\cdot \mid \cdot\rangle)$ over $\mathbb{K}$ is complete

- *if and only if:*
  $$\forall u \in X \quad u = \sum_{k=1}^{\infty} \langle u \mid u_k\rangle \cdot u_k.$$
- *if and only if:*
  $$\forall u \in X \quad \sum_{k=1}^{\infty} |\langle u \mid u_k\rangle|^2 = \|u\|^2,$$

Proposition 380.

<u>*Hyp*</u> *Consider a countably infinite, orthonormal set $\{u_1, u_2, \ldots\} = \{u_k : k \in \mathbb{N}\}$ in a Hilbert space $(X, \langle\cdot \mid \cdot\rangle)$ over $\mathbb{K}$.*
Suppose that the set

$$\left\{\sum_{k=1}^{m} \alpha_k \cdot u_k \;:\; m \in \{1, 2, 3, \ldots\}, \alpha_k \in \mathbb{K} \textit{ for } k = 1, 2, \ldots, m\right\}$$

is dense in X. By this we mean that

$$\forall \varepsilon > 0$$
$$\forall u \in X$$
$$\exists m \in \{1, 2, \ldots\} \textit{ and } \alpha_1, \ldots, \alpha_m \in \mathbb{K} \textit{ with}$$
$$\|u - \textstyle\sum_{k=1}^{m} \alpha_k \cdot u_k\| < \varepsilon.$$

<u>*Concl*</u> *Then this orthonormal set is complete.*

Proof. Let $u \in X$ be given. Then

$$\forall \varepsilon > 0$$
$$\exists m \in \{1, 2, \ldots\} \text{ and } \alpha_1, \ldots, \alpha_m \in \mathbb{K} \text{ with}$$
$$\|u - \textstyle\sum_{k=1}^{m} \alpha_k \cdot u_k\| < \varepsilon.$$

By the last square property, we have

$$\|u - \sum_{k=1}^{m} \langle u \mid u_k\rangle \cdot u_k\| < \varepsilon.$$

Thus

$$u = \sum_{k=1}^{\infty} \langle u \mid u_k\rangle \cdot u_k,$$

so the orthonormal set is complete. □

Parseval's equality

Proposition 381.

<u>*Hyp*</u> *Suppose that the countably infinite, orthonormal set* $\{u_1, u_2, \ldots\} = \{u_k \ : \ k \in \mathbb{N}\}$ *in a Hilbert space* $(X, \langle \cdot \mid \cdot \rangle)$ *over* $\mathbb{K}$ *is complete.*

<u>*Concl*</u> *Then,* $\forall u \in X$ *and* $\forall v \in X$*, we have*

$$\langle u \mid v \rangle = \sum_{k=1}^{\infty} \alpha_k \cdot \overline{\beta_k},$$

where

$$\alpha_k = \langle u \mid u_k \rangle \,, \qquad \text{and} \qquad \beta_k = \langle v \mid u_k \rangle \,, \qquad \text{for } k = 1, 2, \ldots.$$

In short:

$$\boxed{\langle u \mid v \rangle = \sum_{k=1}^{\infty} \langle u \mid u_k \rangle \cdot \overline{\langle v \mid u_k \rangle}.}$$

Part IV

L^2-Fourier theory

12

Fourier series: the L^2-approach

12.1. Fourier series: the classical theory

Let us recall the main result of the classical theory.

Proposition 382.

Hyp Let the T-periodic signal

$$f : \mathbb{R} \to \mathbb{C}, \quad t \mapsto f(t)$$

be of class C^1.
Consider the set $\{c_n\}_{n\in\mathbb{Z}}$ of Fourier coefficients

$$c_n := c_n(f) = \frac{1}{T} \int_0^T f(t) \cdot e^{-2\pi i \frac{n}{T} t}\, dt$$

(representing the contribution of the harmonic of frequency $\frac{k}{T}$ to the given signal f).
Consider the partial sums

$$S_N(f) := \sum_{n=-N}^{N} c_n(f) \cdot e^{2\pi i \frac{n}{T} t}.$$

Concl

1. The partial sums $S_n(f)$ converge to the given signal f:

$$\lim_{N\to\infty} S_N(f) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n(f) \cdot e^{2\pi i \frac{n}{T} t} = f(t), \qquad \forall t \in \mathbb{R}.$$

2. This convergence is uniform:

$$\lim_{N\to\infty} \left(\sup_{t\in\mathbb{R}} |S_N(f)(t) - f(t)| \right) = 0.$$

Thus, $\forall \varepsilon > 0$, there is a threshold $N_0 = N_0(\varepsilon)$ such that

$$\forall N \geq N_0$$
$$|S_N(f)(t) - f(t)| < \varepsilon, \qquad \forall t \in \mathbb{R}.$$

Dini's condition

Proposition 383.

<u>Hyp</u> *Let $f : \mathbb{R} \to \mathbb{C}$ be a piecewise C^1 signal, such that at the points of discontinuities, the unilateral limits of the derivatives exist (Dini's condition).*

<u>Concl</u> *Then*

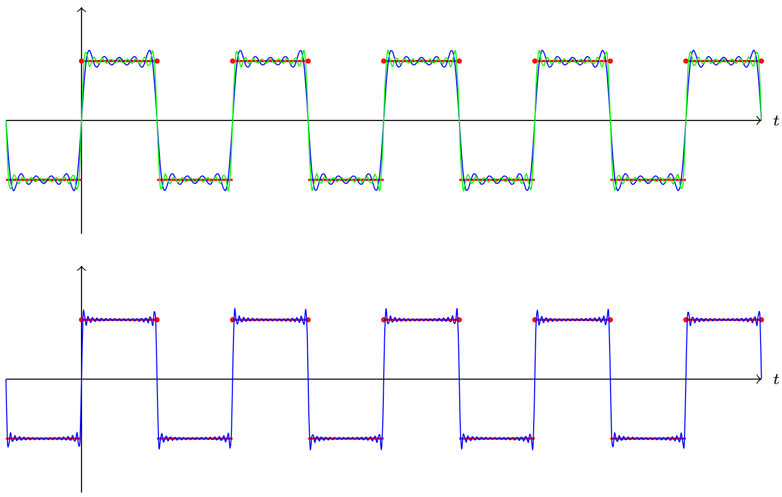
$$\lim_{N\to\infty} (S_N f)(t) = \frac{f(t^-) + f(t^+)}{2}, \qquad \forall t \in \mathbb{R}.$$

Thus, at points of continuity, we have

$$\lim_{N\to\infty} (S_N f)(t) = f(t),$$

whereas at jump points, the limit gives the mean value at the jump.

Remark 384. *Under Dini's condition, the convergence is no longer uniform: at the jump points, one can observe Gibb's phenomenon.*



12.2. A closer look to the formula defining the Fourier coefficients

If one looks at the definition of the Fourier coefficients

$$c_n(f) := \frac{1}{T} \int_0^T f(t) \cdot e^{-2\pi i \frac{n}{T} t}\, dt$$

one may ask for which class of functions these formulas make sense. Since

$$|f(t) \cdot e^{-2\pi i \frac{n}{T} t}| = |f(t)|$$

we can define the Fourier coefficients for all *stable* signals, i.e. for all signals $f \in L^1_{\mathbb{C}}([0,T])$. Thereby

- we interpret $\frac{1}{T}\int_0^T f(t) \cdot e^{-2\pi i \frac{n}{T} t}\, dt$ as a Lebesgue integral and
- we put

$$L^p_{\mathbb{C}}([0,T]) := L^p_{\mathbb{C}}([0,T], \mathscr{L}(\mathbb{R})|_{[0,T]}, \lambda^1|_{[0,T]}).$$

Thus we get

Lemma 385.
For all $f \in L^1_{\mathbb{C}}([0,T])$, the Fourier coefficients

$$c_n(f) := \frac{1}{T}\int_0^T f(t) \cdot e^{-2\pi i \frac{n}{T} t}\, dt$$

(for $n \in \mathbb{Z}$) are all well-defined and finite (i.e. elements in $\mathbb{C}$).

Let us have a second look at the formula defining the Fourier coefficients:

$$c_n(f) := \frac{1}{T}\int_0^T f(t) \cdot e^{-2\pi i \frac{n}{T} t}\, dt.$$

We can write this in the form

$$c_n(f) = \frac{1}{T}\int_0^T f(t) \cdot \overline{e^{2\pi i \frac{n}{T} t}}\, dt. = \frac{1}{\sqrt{T}}\left\langle f(t) \mid \frac{1}{\sqrt{T}} e^{2\pi i \frac{n}{T} t}\right\rangle_{L^2}$$

if the signal f belongs to the Hilbert space $L^2_{\mathbb{C}}([0,T])$ equipped with the inner product

$$\langle u(t) \mid v(t)\rangle_{L^2} = \int_0^T u(t) \cdot \overline{v(t)}\, dt.$$

Remark that, for $n \in \mathbb{Z}$,

$$\left\| \frac{1}{\sqrt{T}} e^{2\pi i \frac{n}{T} t} \right\|_{L^2} = \sqrt{\left\langle \frac{1}{\sqrt{T}} e^{2\pi i \frac{n}{T} t} \mid \frac{1}{\sqrt{T}} e^{2\pi i \frac{n}{T} t}\right\rangle_{L^2}} = 1.$$

Moreover, for $n, m \in \mathbb{Z}$ with $n \neq m$, we have

$$\begin{aligned} \left\langle \frac{1}{\sqrt{T}} e^{2\pi i \frac{n}{T} t} \mid \frac{1}{\sqrt{T}} e^{2\pi i \frac{m}{T} t}\right\rangle_{L^2} &= \frac{1}{T}\int_0^T e^{2\pi i \frac{n-m}{T} t}\, dt \\ &= \frac{1}{T} \cdot \left. \frac{e^{2\pi i \frac{n-m}{T} t}}{2\pi i \frac{n-m}{T}} \right|_0^T = 0. \end{aligned}$$

Thus we get

Lemma 386.
The set

$$\left\{ \frac{1}{\sqrt{T}} \cdot e_\lambda(t) \ : \ \lambda = \frac{n}{T}, n \in \mathbb{Z} \right\},$$

where

$$e_\lambda(t) = e^{2\pi i \lambda t},$$

is a countably infinite, orthonormal set in the Hilbert space $L^2_{\mathbb{C}}([0,T])$.

The sum

$$\sum_{n=-N}^{N} c_n(f) \cdot e^{2\pi i \frac{n}{T} t},$$

where

$$c_n(f) = \frac{1}{T} \int_0^T f(t) \cdot e^{-2\pi i \frac{n}{T} t} \, dt$$

can be written as

$$\begin{aligned} S_N(f)(t) &= \sum_{n=-N}^{N} \left\langle f(t) \mid \frac{1}{\sqrt{T}} \cdot e^{2\pi i \frac{n}{T} t} \right\rangle_{L^2} \cdot \sqrt{T} \cdot e^{2\pi i \frac{n}{T} t} \\ &= \sum_{n=-N}^{N} \left\langle f(t) \mid e_{\frac{n}{T}}(t) \right\rangle_{L^2} \cdot e_{\frac{n}{T}}(t), \end{aligned}$$

with the help of the above introduced countably infinite, orthonormal set

$$\{ \frac{1}{\sqrt{T}} \cdot e_{\frac{n}{T}}(t) \ : \ n \in \mathbb{Z} \}$$

12.3. A dense set in $L^2_{\mathbb{C}}([0,T])$

Proposition 387.
The set

$$A := \{ f : [0,T] \to \mathbb{C} \ : \ f \textit{ is of class } C^1 \textit{ with } f(0) = f(T) \}$$

is dense in $L^2_{\mathbb{C}}([0,T])$*. Thus*

$$\forall \varepsilon > 0, \quad \forall u \in L^2_{\mathbb{C}}([0,T]) \\ \exists v \in A \text{ such that } \quad \|u - v\|_{L^2} < \varepsilon.$$

Proposition 388.

For all signals f *belonging to*

$$A := \{f : [0,T] \to \mathbb{C} \;:\; f \text{ is of class } C^1 \text{ with } f(0) = f(T)\}$$

the corresponding Fourier series converges uniformly to the given signal f*.*

Thus the Fourier series of f *converges to* f *in the* L^2*-norm*

$$\lim_{N\to\infty} \|S_n(f) - f\|_{L^2} = 0.$$

Since

$$A := \{f : [0,T] \to \mathbb{C} \;:\; f \text{ is of class } C^1 \text{ with } f(0) = f(T)\}$$

is dense in $L^2_{\mathbb{C}}([0,T])$, the set

$$\left\{ \sum_{n=-N}^{N} \alpha_n e^{2\pi i \frac{n}{T} t} \;:\; N \in \{1,2,3,\ldots\}, \alpha_n \in \mathbb{C} \text{ for } n = -N, -N+1, \ldots, N \right\}$$

is dense in $L^2_{\mathbb{C}}([0,T])$, too. Thus we get

Proposition 389.

The countably infinite, orthonormal set

$$\{\frac{1}{\sqrt{T}} \cdot e_{\frac{n}{T}}(t) \;:\; n \in \mathbb{Z}\}$$

is complete in $L^2_{\mathbb{C}}([0,T])$*. Thus*

$$\forall f \in L^2_{\mathbb{C}}([0,T]), \quad \lim_{N\to\infty} \|S_N(f) - f\|_{L^2} = 0.$$

Proof. This follows from

$$\begin{aligned} f &= \lim_{N\to\infty} \sum_{n=-N}^{N} \left\langle f(t) \mid \frac{1}{\sqrt{T}} \cdot e^{2\pi i \frac{n}{T} t} \right\rangle \cdot \sqrt{T} \cdot e^{2\pi i \frac{n}{T} t} \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} c_n(f) \cdot e^{2\pi i \frac{n}{T} t}. \end{aligned}$$

□

Thus we get

$$\boxed{f = \sum_{n\in\mathbb{Z}} c_n(f)\cdot e^{2\pi i \frac{n}{T} t}, \qquad f \in L^2_{\mathbb{C}}([0,T]),}$$

where the above equality must be interpreted as an eqnality in $L^2_{\mathbb{C}}([0,T])$.

12.4. The L^2-theory for Fourier series

We get in this way a mapping

$$\mathscr{F}_T : L^2_{\mathbb{C}}([0,T]) \to \ell^2_{\mathbb{C}}(\mathbb{Z}), \qquad f(t) \mapsto \{c_n(f)\}_{n\in\mathbb{Z}},$$

where $\ell_{\mathbb{C}}(\mathbb{Z})$ is the Hilbert space including all "double-sided" sequences

$$\dots, c_{-n}, \dots, c_{-1}, c_0, c_1, \dots, c_n, \dots$$

with

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = \lim_{n\to\infty} \sum_{n=-N}^{N} |c_n|^2 < +\infty$$

equipped with the scalar product

$$\langle \{a_n\} \mid \{b_n\}\rangle_{\ell^2} = \lim_{N\to\infty} \sum_{n=-N}^{N} a_n \cdot \overline{b_n} = \sum_{n\in\mathbb{Z}} a_n \cdot \overline{b_n}.$$

Indeed, due to Bessel's equality, we have, for all $f \in L^2_{\mathbb{C}}([0,T])$,

$$\begin{aligned}
\|f\|^2_{L^2} &= \lim_{N\to\infty} \sum_{n=-N}^{N} \left| \left\langle f(t) \mid \frac{1}{\sqrt{T}} \cdot e^{2\pi i \frac{n}{T} t} \right\rangle_{L^2} \right|^2 \\
&= T \cdot \lim_{N\to\infty} \sum_{n=-N}^{N} |c_n(f)|^2 \\
&= T \cdot \|\{c_n(f)\|^2_{\ell^2} < +\infty.
\end{aligned}$$

Moreover, due to Parseval's equality, we have, for all $f, g \in L^2_{\mathbb{C}}([0,T])$,

$$\begin{aligned}
\langle f \mid g\rangle_{L^2} &= \lim_{N\to\infty} \sum_{n=-N}^{N} \left\langle f(t) \mid \frac{1}{\sqrt{T}} \cdot e^{2\pi i \frac{n}{T} t} \right\rangle_{L^2} \cdot \overline{\left\langle g(t) \mid \frac{1}{\sqrt{T}} \cdot e^{2\pi i \frac{n}{T} t} \right\rangle_{L^2}} \\
&= T \cdot \lim_{N\to\infty} \sum_{n=-N}^{N} c_n(f) \cdot \overline{c_n(g)} = T \cdot \sum_{n\in\mathbb{Z}} c_n(f) \cdot \overline{c_n(g)} \\
&= T \cdot \langle \{c_n(f)\} \mid \{c_n(g)\}\rangle_{\ell^2}.
\end{aligned}$$

Putting this all together, we get

Proposition 390.
The mapping

$$\mathscr{F}_T : L^2_{\mathbb{C}}([0,T]) \to \ell^2_{\mathbb{C}}(\mathbb{Z}), \qquad f(t) \mapsto \{c_n(f)\}_{n\in\mathbb{Z}},$$

is a well-defined bijection, with

1. $\|f\|^2_{L^2} = T \cdot \|\{c_n(f)\}\|^2_{\ell^2}$, $\forall f \in L_{\mathbb{C}}([0,T])$;
2. $\langle f \mid g\rangle_{L^2} = T \cdot \langle\{c_n(f)\} \mid \{c_n(f)\}\rangle_{\ell^2}$, $\forall f, g \in L_{\mathbb{C}}([0,T])$;
3. $\mathscr{F}_T$ *is linear:*

$$c_n(\alpha \cdot f + g) = \alpha \cdot c_n(f) + c_n(g), \qquad \forall \alpha \in \mathbb{C}, \quad \forall f, g \in L^2_{\mathbb{C}}([0,T]).$$

4. *The partial sums*

$$S_N(f) := \sum_{n=-N}^{N} c_n(f) \cdot e^{2\pi i \frac{n}{T} t}$$

are the best possible approximation of the given signal $f \in L^2_{\mathbb{C}}([0,T])$ *with respect to all signals inside*

$$\text{span}\ \left\{e^{2\pi i \frac{n}{T} t}\ :\ n \in \{-N, \dots, -1, 0, 1, \dots, N\}\right\}.$$

13

Fourier transform: the L^2-approach

13.1. The Fourier transform in L^1

We put in what follows
$$L^p_{\mathbb{C}}(\mathbb{R}) := L^p_{\mathbb{C}}(\mathbb{R}, \mathscr{L}(\mathbb{R}), \lambda^1)$$
and we interpret integrals like
$$\int_{\mathbb{R}} f(t)\, dt$$
as Lebesgue integrals.

The we have yet mentioned the following result.

Proposition 391.
The Fourier transform

$$\begin{aligned} &\mathscr{F}_{L^1} : L^1_{\mathbb{C}}(\mathbb{R}) \to C_b(\mathbb{R}), \\ &f(t) \mapsto \mathscr{F}_{L^1}[f(t)](\lambda) := \int_{\mathbb{R}} f(t) \cdot e^{2\pi i \lambda t}\, dt = \hat{f}(\lambda), \end{aligned}$$

where $C_b(\mathbb{R})$ is the set containing all continuous functions $f : \mathbb{R} \to \mathbb{C}$ with

$$\|f\|_\infty := \sup_{t\in\mathbb{R}} |f(t)| < +\infty,$$

is a well-define, bounded and linear operator with

$$\|\mathscr{F}_{L^1}\| \le 1,$$

i.e. with
$$\sup_{\lambda\in\mathbb{R}} |\hat{f}(\lambda)| \le \|f\|_{L^1},$$

Remark 392. *Let us mention that*

$$C_b(\mathbb{R}) \subset L^\infty_{\mathbb{C}}(\mathbb{R})$$

and that
$$\|f\|_\infty = \|f\|_{L^\infty}, \qquad \forall f \in C_b(\mathbb{R}).$$

Thus we may consider $\mathscr{F}_{L^1}$ as a linear, bounded mapping

$$\mathscr{F}_{L^1} : L^1_{\mathbb{C}}(\mathbb{R}) \to L^\infty_{\mathbb{C}}(\mathbb{R}).$$

Here again
$$\|\mathscr{F}_{L^1}\| \le 1,$$

i.e.
$$\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1}, \qquad \forall f \in L^1_{\mathbb{C}}(\mathbb{R}).$$

Lemma of Riemann-Lebesgue

Proposition 393.
For all $f \in L^1_{\mathbb{C}}(\mathbb{R})$, we have

$$\lim_{\lambda\to\pm\infty} \hat{f}(\lambda) = 0.$$

Proof. This is so for $f(x) = \chi_{[a,b]}$ since in this case

$$\hat{f}(\lambda) = \int_a^b e^{2\pi i\lambda t}\, dt = \left.\frac{e^{2\pi i\lambda t}}{2\pi i\lambda}\right|_a^b$$

so that

$$|\hat{f}(\lambda)| \leq \frac{1}{\pi|\lambda|}, \qquad \text{for } \lambda \neq 0.$$

Thus, the conclusion of the proposition is verified for all simple functions $f \in \mathscr{T}_{\mathbb{C}}(\mathbb{R}, \mathscr{L}(\mathbb{R}))$.

The general case follows now by remarking that, given a $f \in L^1_{\mathbb{C}}(\mathbb{R})$, there exists a sequence of simple functions $\{g_n\}_{n=1}^{+\infty}$ with

$$\lim_{n\to\infty} \|f - g_n\|_{L^1} = 0.$$

Thus, for any sequence $\{\lambda_n\}_{n=1}^{+\infty}$ converging to $+\infty$ or to $-\infty$, we have

$$|f(\lambda_n) - g_n(\lambda_n)| \leq \|f - g_n\|_{L^1} \to 0, \qquad \forall\lambda \in \mathbb{R}.$$

This implies

$$\lim_{\lambda\to\pm\infty} \hat{f}(\lambda) = 0$$

since $\lim_{\lambda\to\pm\infty} \hat{g_n}(\lambda) = 0$. □

Proposition 394.

<u>Hyp</u> *Let f and g be two signals in $L^1_{\mathbb{C}}(\mathbb{R})$.*
<u>Concl</u> *Then*

1. *Both $f(t)\cdot\hat{g}(t)$ and $\hat{f}(t)\cdot g(t)$ belong to $L^1_{\mathbb{C}}(\mathbb{R})$.*
2. *Moreover*

$$\int_{\mathbb{R}} f(t)\cdot\hat{g}(t)\, dt = \int_{\mathbb{R}} \hat{f}(\lambda)\cdot g(\lambda)\, d\lambda.$$

Proof. The first point follows by Hölder from $\hat{f}, \hat{g} \in L^\infty_{\mathbb{C}}(\mathbb{R})$.

Concerning the second point, we have by Fubini's theorem, since

$$f(t)g(\lambda)e^{-2\pi i\lambda t} \in L^1_{\mathbb{C}}(\mathbb{R}^2),$$

that

$$\begin{aligned}\int_{\mathbb{R}} \hat{f}(\lambda)g(\lambda)\, dt &= \int_{\mathbb{R}} g(\lambda) \int_{\mathbb{R}} f(t)e^{-2\pi i\lambda t}\, dt\, d\lambda \\ &= \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} g(\lambda)e^{-2\pi i\lambda t}\, d\lambda\, dt \\ &= \int_{\mathbb{R}} f(t) \cdot \hat{g}(t)\, dt.\end{aligned}$$

□

13.2. Rules for computing with the Fourier transform $\mathscr{F}_{L^1}$

13.2.1. Linearity

$\alpha \cdot f(t) + g(t) \quad \circ\!\!-\!\!- \quad \alpha \cdot \hat{f}(\lambda) + \hat{g}(\lambda)$

Proposition 395.

The Fourier transform $\mathscr{F}_{L^1} : L^1_{\mathbb{C}}(\mathbb{R}) \to L^\infty_{\mathbb{C}}(\mathbb{R})$ is linear.

Thus, $\forall \alpha \in \mathbb{C}$, $\forall f, g \in L^1_{\mathbb{C}}(\mathbb{R})$,

$$\mathscr{F}_{L^1}[\alpha \cdot f(t) + g(t)](\lambda) = \alpha \cdot \mathscr{F}_{L^1}[f(t)](\lambda) + \mathscr{F}_{L^1}[g(t)](\lambda),$$

i.e.

$$[\alpha \cdot f(t) + g(t)]^\wedge = \alpha \cdot \hat{f}(\lambda) + \hat{g}(\lambda).$$

Proof. This follows from

$$\begin{aligned}\int_{\mathbb{R}} (\alpha \cdot f(t) + g(t)) \cdot e^{-2\pi i\lambda t}\, dt &= \int_{\mathbb{R}} \left(\alpha \cdot f(t) \cdot e^{-2\pi i\lambda t} + g(t) \cdot e^{-2\pi i\lambda t}\right)\, dt \\ &= \alpha \cdot \int_{\mathbb{R}} f(t) \cdot e^{-2\pi i\lambda t}\, dt + \int_{\mathbb{R}} g(t) \cdot e^{-2\pi i\lambda t}\, dt\end{aligned}$$

□

13.2.2. Multiplication by powers of time

$(-2\pi i t)^m f(t) \quad \circ\!\!-\!\!- \quad \frac{d^m}{d\lambda^m}\hat{f}(\lambda)$

Proposition 396.

<u>*Hyp*</u> *Suppose that the signal f is such that*

$$t^k \cdot f(t) \in L^1_{\mathbb{C}}(\mathbb{R}), \qquad \text{for } k = 0, 1, 2, \ldots, m$$

<u>*Concl*</u> *Then its Fourier transformed $\hat{f}(\lambda)$ is m times differentiable and, for $k = 0, 1, 2, \ldots, m$,*

$$\frac{d}{d\lambda^k}\hat{f}(\lambda) = \mathscr{F}_{L^1}[(-2\pi i t)^k \cdot f(t)](\lambda) = [(-2\pi i t)^k \cdot f(t)]^\wedge(\lambda).$$

Proof. We give the proof for $m = 1$. For $m > 1$, one can use induction.

$$\begin{aligned}\frac{d}{d\lambda}\hat{f}(\lambda) &= \lim_{h\to 0}\frac{\hat{f}(\lambda+h)-\hat{f}(\lambda)}{h}\\ &= \lim_{h\to 0}\int_{\mathbb{R}} f(t)\cdot\frac{e^{-2\pi i(\lambda+h)t}-e^{-2\pi i\lambda t}}{h}\,dt\end{aligned}$$

In order to apply Legesgue's dominated convergence theorem, we need an integrable majoration for the integrand. But

$$f(t)\cdot\frac{e^{-2\pi i(\lambda+h)t}-e^{-2\pi i(\lambda+h)t}}{h} = f(t)\cdot e^{-2\pi i(\lambda+\vartheta\cdot h)t}\cdot(-2\pi i t)$$

for some $\vartheta \in]0,1[$.

Thus we get the majoration

$$\left|f(t)\cdot\frac{e^{-2\pi i(\lambda+h)t}-e^{-2\pi i(\lambda+h)t}}{h}\right| = |t\cdot f(t)| \in L^1_{\mathbb{C}}(\mathbb{R}).$$

By dominated convergence, we get now

$$\begin{aligned}\frac{d}{d\lambda}\hat{f}(\lambda) &= \lim_{h\to 0}\int_{\mathbb{R}} f(t)\cdot\frac{e^{-2\pi i(\lambda+h)t}-e^{-2\pi i\lambda t}}{h}\,dt\\ &= \lim_{h\to 0}\int_{\mathbb{R}}\lim_{h\to 0}\left(f(t)\cdot\frac{e^{-2\pi i(\lambda+h)t}-e^{-2\pi i\lambda t}}{h}\right)\,dt\\ &= \int_{\mathbb{R}}(-2\pi i t)\cdot f(t)\cdot e^{-2\pi i\lambda t}\,dt\\ &= \mathscr{F}_{L^1}[(-2\pi i t)\cdot f(t)](\lambda).\end{aligned}$$

□

13.2.3. Derivatives

$\frac{d^m}{dt^m} f(t) \quad \circ\!\!-\!\!- \quad (2\pi i\lambda)^m \cdot \hat{f}(\lambda)$

Proposition 397.

Hyp Suppose that the signal f is such that

- f is of class C^n for some $n \geq 1$;
- $f, f', \ldots, f^{(n)} \in L^1_{\mathbb{C}}(\mathbb{R})$.

Concl Then, for $k = 1, 2, \ldots, n$,

$$\mathscr{F}_{L^1}[f^k(t)](\lambda) = (2\pi i\lambda)^k \cdot \mathscr{F}_{L^1}[f(t)](\lambda),$$

i.e.

$$\widehat{f^{(k)}}(\lambda) = (2\pi i\lambda)^k \cdot \hat{f}(\lambda).$$

Proof. We give the proof for $n = 1$. For $n > 1$, the result follows by induction.

Since $f' \in L^1_{\mathbb{C}}(\mathbb{R})$, we have (by dominated convergence) that

$$\begin{aligned}
\widehat{f'}(\lambda) &= \lim_{\tau\to\pm\infty} \int_{-\tau}^{\tau} f'(t) \cdot e^{-2\pi i\lambda t}\, dt \\
&= \lim_{\tau\to\pm\infty} f(t) \cdot e^{2\pi i\lambda t}\Big|_{-\tau}^{\tau} + \lim_{\tau\to\pm\infty} \int_{-\tau}^{\tau} (2\pi i\lambda) \cdot f(t) \cdot e^{2\pi i\lambda t}\, dt \\
&= \lim_{\tau\to\pm\infty} f(t) \cdot e^{2\pi i\lambda t}\Big|_{-\tau}^{\tau} + \int_{\mathbb{R}} (2\pi i\lambda) \cdot f(t) \cdot e^{2\pi i\lambda t}\, dt \\
&= \lim_{\tau\to\pm\infty} f(t) \cdot e^{2\pi i\lambda t}\Big|_{-\tau}^{\tau} + (2\pi i\lambda) \cdot \hat{f}(\lambda).
\end{aligned}$$

Thus, if we can show that

$$\lim_{\tau\to-\infty} f(t) = \lim_{\tau\to+\infty} f(t) = 0.$$

we are done!

Since $f \in L^1_{\mathbb{C}}(\mathbb{R})$, it is enough to show that the limits

$$\lim_{\tau\to-\infty} f(t) \qquad \text{and} \qquad \lim_{\tau\to+\infty} f(t)$$

both exist. We show this for $\tau \to +\infty$ and leave it to the reader, to check in a similar way the result for $\tau \to -\infty$.

We have

$$f(\tau) = f(0) + \int_o^\tau f'(t)\, dt$$

an thus, since $f' \in L^1_{\mathbb{C}}(\mathbb{R})$, the following limit exists:

$$\lim_{\tau\to+\infty} f(\tau) = f(0) + \int_0^{+\infty} f'(t)\, dt.$$

□

13.2.4. Shift in time

$f(t-a) \quad \circ\!\!-\!\!- \quad e^{-2\pi i\lambda a} \cdot \hat{f}(\lambda)$

Proposition 398.

<u>*Hyp*</u> *Let $f \in L^1_{\mathbb{C}}(\mathbb{R})$ and $a \in \mathbb{R}$ be fixed and given.*

<u>*Concl*</u> *Then*

$$\mathscr{F}_{L^1}[f(t-a)](\lambda) = e^{-2\pi i\lambda a} \cdot \mathscr{F}_{L^1}[f(t)](\lambda),$$

i.e.

$$\widehat{f(t-a)}(\lambda) = e^{-2\pi i\lambda a} \cdot \hat{f}(\lambda).$$

Proof. This follows immediately form

$$\begin{aligned}\int_{\mathbb{R}} f(t-a) \cdot e^{-2\pi i\lambda t}\, dt &= \int_{\mathbb{R}} f(t) \cdot e^{-2\pi i\lambda(t+a)}\, dt \\ &= e^{-2\pi i\lambda a} \cdot \int_{\mathbb{R}} f(t) \cdot e^{-2\pi i\lambda t}\, dt,\end{aligned}$$

□

13.2.5. Modulation

$e^{2\pi i\omega t} \cdot f(t) \quad \circ\!\!-\!\!- \quad \hat{f}(\lambda-\omega)$

Proposition 399.

<u>*Hyp*</u> *Let $f \in L^1_{\mathbb{C}}(\mathbb{R})$ and $\omega \in \mathbb{R}$ be fixed and given.*

<u>Concl</u> *Then*

$$\mathscr{F}_{L^1}[e^{2\pi i\omega t} \cdot f(t)](\lambda) = \mathscr{F}_{L^1}[f(t)](\lambda - \omega),$$

i.e.

$$\widehat{e^{2\pi i\omega t} \cdot f}(t)(\lambda) = \hat{f}(\lambda - \omega).$$

Proof. The claim follows from

$$\int_{\mathbb{R}} e^{2\pi i\omega t} \cdot f(t) \cdot e^{-2\pi i\lambda t} \, dt \quad = \quad \int_{\mathbb{R}} f(t) \cdot e^{-2\pi i(\lambda-\omega)t} \, dt$$

□

13.2.6. Scaling

$f(a \cdot t) \quad \circ\!\!-\!\!- \quad \frac{1}{|a|} \cdot \hat{f}(\lambda/a)$

Proposition 400.

<u>Hyp</u> *Let $f \in L^1_{\mathbb{C}}(\mathbb{R})$ and $a \in \mathbb{R} \setminus \{0\}$ be fixed and given.*
<u>Concl</u> *Then $f(a \cdot t) \in L^1_{\mathbb{C}}(\mathbb{R})$ and*

$$\mathscr{F}_{L^1}[f(a \cdot t)](\lambda) = \frac{1}{|a|} \cdot \mathscr{F}_{L^1}[f(t)] \left(\frac{\lambda}{a}\right),$$

i.e.

$$\widehat{f(a \cdot t)}(\lambda) = \frac{1}{|a|} \cdot \hat{f}\left(\frac{\lambda}{a}\right).$$

Proof. This follows form

$$\int_{\mathbb{R}} f(a \cdot t) e^{-2\pi i\lambda t} \, dt \quad = \quad \frac{1}{|a|} \cdot \int_{\mathbb{R}} f(t) \cdot e^{-2\pi i\lambda t/a} \, dt.$$

□

13.2.7. Convolution

Proposition 401.

<u>Hyp</u> *Suppose that $f \in L^1_{\mathbb{C}}(\mathbb{R})$ and that $g \in L^p_{\mathbb{C}}(\mathbb{R})$ with $p \in [1, +\infty[$.*

Concl *The convolution*

$$(f * g)(t) := \int_{\mathbb{R}} f(t-\tau) \cdot g(\tau)\, d\tau$$

exists for a.a. $t \in \mathbb{R}$.
Moreover

$$(f * g) \in L^p_{\mathbb{C}}(\mathbb{R}).$$

and

$$\|f * g\|_p \leq \|f\|_1 \cdot \|g\|_p.$$

$(f * g)(t) \quad \circ\!\!-\!\!- \hat{f}(\lambda) \cdot \hat{g}(\lambda)$

Proposition 402.

Hyp *Suppose that* $f, g \in L^1_{\mathbb{C}}(\mathbb{R})$, *so that* $f * g \in L^1_{\mathbb{C}}(\mathbb{R})$ *again.*
Concl *Then*

$$\mathscr{F}_{L^1}[(f * g)(t)](\lambda) = \mathscr{F}_{L^1}[f(t)](\lambda) \cdot \mathscr{F}_{L^1}[g(t)](\lambda),$$

i.e.

$$\widehat{f * g}(\lambda) = \hat{f}(\lambda) \cdot \hat{g}(\lambda).$$

Proof. Since $f * g \in L^1_{\mathbb{C}}(\mathbb{R})$, we may use Fubini's theorem to get

$$\begin{aligned}\int_{\mathbb{R}} (f * g)(t) e^{-2\pi i \lambda t}\, dt &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t-\tau) \cdot g(\tau)\, d\tau \right) e^{-2\pi i \lambda t}\, dt \\ &= \int_{\mathbb{R}} g(\tau) \cdot e^{-2\pi i \lambda \tau} \int_{\mathbb{R}} f(t-\tau) \cdot e^{-2\pi i \lambda (t-\tau)}\, dt\, d\tau \\ &= \int_{\mathbb{R}} g(\tau) \cdot e^{-2\pi i \lambda \tau} \underbrace{\left(\int_{\mathbb{R}} f(t) \cdot e^{-2\pi i \lambda t}\, dt \right)}_{=\hat{f}(\lambda)} d\tau \\ &= \hat{f}(\lambda) \cdot \int_{\mathbb{R}} g(\tau) \cdot e^{-2\pi i \lambda \tau}\, d\tau = \hat{f}(\lambda) \cdot \hat{g}(\lambda).\end{aligned}$$

□

13.3. The inverse Fourier transform in L^1

Definition 403.
The inverse Fourier transform is defined by

$$\mathscr{F}_{L^1}^{-1} : L^1_{\mathbb{C}}(\mathbb{R}) \to L^\infty_{\mathbb{C}}(\mathbb{R}), \quad f(\lambda) \mapsto \mathscr{F}_{L^1}^{-1}[f(\lambda)](t) = \int_{\mathbb{R}} f(\lambda)e^{2\pi i\lambda t}\, d\lambda.$$

Remark 404. *The inverse Fourier transform $\mathscr{F}_{L^1}^{-1}$ has the similar properties as $\mathscr{F}_{L^1}$:*

- *linearity:*
$$\alpha \cdot \hat{f}(\lambda) + \hat{g}(\lambda) \quad \multimap \quad \alpha \cdot f(t) + g(t)$$
- *multiplication by powers of λ and derivatives:*
$$(2\pi i\lambda)^m \cdot \hat{f}(\lambda) \quad \circ\!\!- \quad \frac{d^m}{dt^m} f(t), \qquad \frac{d^m}{d\lambda^m}\hat{f}(\lambda) \quad \circ\!\!- \quad (-2\pi it)^m f(t)$$
- *shift in λ and modulation in time:*
$$\hat{f}(\lambda - \omega) \quad \circ\!\!- \quad e^{2\pi i\omega t} \cdot f(t), \qquad e^{-2\pi i\lambda a} \cdot \hat{f}(\lambda) \quad \circ\!\!- \quad f(t-a)$$
- *scaling:*
$$\frac{1}{|a|} \cdot \hat{f}(\lambda/a) \quad \circ\!\!- \quad f(a \cdot t).$$

Proposition 405.

Hyp *Suppose that the signal f and its Fourier transform $\hat{f}(\lambda)$ both belong to $L^1_{\mathbb{C}}(\mathbb{R})$.*

Concl *Then*
$$\mathscr{F}_{L^1}^{-1}[\hat{f}(\lambda)](t) = f(t)$$
in all points t where f is continuous.

Proof. We will omit the somewhat long proof! □

Remark 406. *Remark that the signal $\chi_{[a,b]}(t)$ (rectangular pulse) belongs to $L^1_{\mathbb{C}}(\mathbb{R})$, but its Fourier transform does* not *belong to $L^1_{\mathbb{C}}(\mathbb{R})$.*

This simple example shows the limitation of the above result.

13.4. The Schwarz space

Definition 407.
The Schwarz space is the set of all functions

$$f : \mathbb{R} \to \mathbb{C}, \quad t \mapsto f(t)$$

satisfying the following properties;

1. $f \in C^\infty(\mathbb{R})$;
2. for all n and $m \in \{0, 1, 2, 3, \ldots\}$ we have

$$\lim_{t \to \pm\infty} t^n \cdot f^{(m)}(t) = 0$$

(i.e. f is quickly decreasing!).

We denote this space by $\mathscr{S}$.

Proposition 408.
We have

$$\mathscr{S} \subset L^p_{\mathbb{C}}(\mathbb{R}), \qquad \forall p \in [1, \infty[,$$

and the above inclusion is dense. Thus

$$\forall f \in L^p_{\mathbb{C}}(\mathbb{R}), \quad \forall \varepsilon > 0$$
$$\exists g \in \mathscr{S} \text{ such that } \|f - g\|_{L^p} < \varepsilon.$$

Remark 409. *Thus, the Schwarz space $\mathscr{S}$ is quite large.*
Moreover, $\mathscr{F}_{L^1}[f(t)](\lambda)$ is well-defined for all signals $f \in \mathscr{S}$.

It follows immediately from the above definition, that the Schwarz space is closed under "taking derivatives" and "multiplying with t":

Proposition 410.

Hyp *Suppose that the signal f belongs to the Schwarz space $\mathscr{S}$*

Concl *Then*

1. $t^n \cdot f(t) \in \mathscr{S}, \quad \forall n = 1, 2, 3, \ldots;$
2. $f^{(m)}(t) \in \mathscr{S}, \quad \forall m = 1, 2, 3, \ldots.$

Proposition 411.
If one restrict the Fourier transform $\mathscr{F}_{L^1}$ to the Schwarz space, i.e. if one considers

$$\mathscr{F} := \mathscr{F}_{L^1}|_{\mathscr{S}},$$

then

$$\mathscr{F} : \mathscr{S} \to \mathscr{S}, \qquad f(t) \mapsto \mathscr{F}[f(t)](\lambda) := \int_{\mathbb{R}} f(t) e^{2\pi i \lambda t}\, dt$$

is a linear mapping, that hat the same properties as $\mathscr{F}_{L^1}$ with respect to

- *linearity,*
- *multiplication by a power of time,*
- *differentiability,*
- *shift in time,*
- *modulation,*
- *convolution.*

Moreover, $f \in \mathscr{S} \Longrightarrow \mathscr{F}[f(t)](\lambda) \in \mathscr{S}$.

Proof. Let us fix some $f \in S$. Then

$$t^n \cdot f(t) \in \mathscr{S} \subset L^1_{\mathbb{C}}(\mathbb{R}), \qquad \forall n \in \{1, 2, 3, \ldots\}.$$

Thus $\hat{f}(\lambda)$ is of class C^∞.

Moreover, the signal

$$g(t) := \frac{d^m}{dt^m}\left[(-2\pi i t)^n \cdot f(t)\right]$$

belongs to the Schwarz space $\mathscr{S}$ ($m, n \in \{0, 1, 2, 3, \ldots\}$); thus $g \in L^1_{\mathbb{C}}(\mathbb{R})$, so that, due to the Lemma of Lebesgue-Riemann (see 393)

$$\lim_{\lambda \to \pm\infty} \hat{g}(\lambda) = 0.$$

But

$$\begin{aligned} \hat{g}(\lambda) &= (2\pi i \lambda)^m \mathscr{F}[(-2\pi i t)^n \cdot f(t)](\lambda) \\ &= (2\pi i \lambda)^m \frac{d^n}{d\lambda^n} \hat{f}(\lambda), \end{aligned}$$

so that

$$\lim_{\lambda \to \pm\infty} \lambda^m \frac{d^n}{d\lambda^n} \hat{f}(\lambda) = 0$$

Thus

$$f \in \mathscr{S} \Longrightarrow \mathscr{F}[f(t)](\lambda) \in \mathscr{S}.$$

Since $\mathscr{F}$ is a restriction of $\mathscr{F}_{L^1}$, this mapping $\mathscr{F}$ has the same properties as $\mathscr{F}_{L^1}$. So we are done! □

Remark that, for all $f \in \mathscr{S}$, we have

1. $\hat{f} \in \mathscr{S} \subset L^1_{\mathbb{C}}(\mathbb{R})$, so $\mathscr{F}^{-1}_{L^1}[\hat{f}(\lambda)](t)$ is well-defined and
2. since f is continuous

$$f(t) = \mathscr{F}^{-1}_{L^1}[\hat{f}(\lambda)](t), \qquad \forall t \in \mathbb{R}.$$

Remark however, that

$$\begin{aligned} f(t) &= \mathscr{F}^{-1}[\hat{f}(\lambda)](t) \\ &= \int_{\mathbb{R}} \hat{f}(\lambda) e^{2\pi i \lambda t}\, d\lambda \\ &= \mathscr{F}[\hat{f}(\lambda)](-t) \\ f(-t) &= \mathscr{F}[\hat{f}(\lambda)](t), \qquad \forall t \in \mathbb{R}. \end{aligned}$$

Thus we get

$$\boxed{\mathscr{F}^2[f(\tau)](t) = f(-t), \qquad \forall t \in \mathbb{R}}$$

Proposition 412.

Hyp *Let*

$$\mathscr{F} : \mathscr{S} \to \mathscr{S}$$

be the restriction of $\mathscr{F}_{L^1}$ *to the schwarz space.*

Concl

1. $$\boxed{\mathscr{F}^4[f(\tau)](t) = f(t), \qquad t \in \mathbb{R}.}$$

2. $$\boxed{\mathscr{F}^3 \text{ is the inverse mapping } \mathscr{F}^{-1} \text{ to } \mathscr{F}}.$$

   Thus

   - $\mathscr{F} \circ \mathscr{F}^{-1} = \mathbb{I}$ *on* $\mathscr{S}$;
   - $\mathscr{F}^{-1} \circ \mathscr{F} = \mathbb{I}$ *on* $\mathscr{S}$

3. *The mapping* $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ *is a bijection.*

Proof. All follows from

$$\mathscr{F} \circ \mathscr{F}^{-1} = \mathscr{F} \circ \mathscr{F}^3 = \mathscr{F}^4 = \mathbb{I}$$

and

$$\mathscr{F}^{-1} \circ \mathscr{F} = \mathscr{F}^3 \circ \mathscr{F} = \mathscr{F}^4 = \mathbb{I}.$$

□

$\mathscr{F} : \mathscr{S} \to \mathscr{S}$ preserves the L^2-norm

Proposition 413.
The bijection

$$\mathscr{F} : \mathscr{S} \to \mathscr{S}$$

preserves the L^2*-norm and the* L^2*-inner product. Thus, for all* f *and* $g \in \mathscr{S}$*, we have*

$$\|\mathscr{F}[f(t)](\lambda)\|_{L^2} = \|f(t)\|_{L^2}$$

and

$$\left\langle \hat{f}(\lambda) \mid \hat{g}(\lambda) \right\rangle_{L^2} = \langle f(t) \mid g(t)_{L^2} \rangle .$$

Thus, $\mathscr{F}$ *is a linear, bounded operator that preserves the* L^2*-norm:*

$$\mathscr{F} \in L((\mathscr{S}, \| \cdot \|_{L^2})).$$

Remark however that $(\mathscr{S}, \| \cdot \|_{L^2})$ *is a pre-hilbert space, but it is not a Hilbert space.*

Remark 414. *Remark that* $\langle f(t) \mid \underline{g(t)_{L^2}} \rangle$ *is well-defined for* f *and* $g \in \mathscr{S}$*, since then* f *and* $g \in L^2_{\mathbb{C}}(\mathbb{R})$*, so that the product* $f(t) \cdot \overline{g(t)}$ *is integrable.*

The same argument shows that

$$\left\langle \hat{f}(\lambda) \mid \hat{g}(\lambda) \right\rangle_{L^2}, \quad \|\mathscr{F}[f(t)](\lambda)\|_{L^2} \quad \textit{and} \quad \|f(t)\|_{L^2}$$

are all well-defined.

13.5. The Fourier transform in L^2

13.5.1. Densly defined bounded, linear operators

Proposition 415.

Hyp — *Suppose that*

- $(X, \|\cdot\|_X)$ *is a normed space and that*
- $(Y, \|\cdot\|_Y)$ *is a Banach space.*

Let $D \subset X$ *be a dense, linear subspace and consider a bounded and linear operator*

$$T : D \to Y.$$

Concl — *Then, there exists a unique bounded and linear extension* $\tilde{T}$ *of* T *to* X*:*

$$\tilde{T} : X \to Y, \qquad \tilde{T}|_D = T.$$

Thus

$$Tx = \tilde{T}x, \qquad \forall x \in X.$$

Moreover, we have

$$\|T\| = \|\tilde{T}\|.$$

Proof. D is dense in X. Thus, for each $x \in X$, we can choose a sequence $\{x_n\}_{n=1}^{+\infty}$ in D in such a way that

$$\lim_{n\to\infty} x_n = x.$$

(I) The image sequence $\{y_n\}_{n=1}^{+\infty}$ in Y defined by $y_n := Tx_n$ is Cauchy:
This follows from

$$\|y_n - y_m\| = \|Tx_n - Tx_m\| = \|T(x_n - x_m)| \le \|T\| \cdot \|x_n - x_m\|.$$

(II) Definition of the extension:
We put now

$$\tilde{T}x := \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n.$$

Remark that $\tilde{T}$ is well defined, since the value $\tilde{T}x$ does not depend on the chosen sequence $\{x_n\}_{n=1}^{+\infty}$ converging to x.

Indeed

$$\left.\begin{array}{l} x_n \to x, Tx_n \to y \\ \bar{x}_n \to x, T\bar{x}_n \to \bar{y} \end{array}\right\} \Longrightarrow \|y-\bar{y}\| = \lim_{n\to\infty} \underbrace{\|T(x_n-\bar{x}_n)\|}_{\leq \|T\|\cdot\|x_n-\bar{x}_n\|\to 0} = 0 \Longrightarrow y = \bar{y}.$$

(II) The extension $\tilde{T}$ is linear:
Indeed

$$\left.\begin{array}{l} x_n \to x \\ \bar{x}_n \to \bar{x} \end{array}\right\} \Longrightarrow \alpha\cdot x_n + \bar{x}_n \to \alpha\cdot x + \bar{x} \Longrightarrow T(\alpha\cdot x_n + \bar{x}_n) \to \tilde{T}(\alpha\cdot x + \bar{x})$$

But

$$T(\alpha\cdot x_n + \bar{x}_n) = \alpha\cdot Tx_n + T\bar{x}_n \to \alpha\cdot\tilde{T}x + \tilde{T}\bar{x}.$$

Thus

$$\tilde{T}(\alpha\cdot x + \bar{x}) = \alpha\cdot\tilde{T}x + \tilde{T}\bar{x},$$

so $\tilde{T}$ is linear.
(III) $\|\tilde{T}\| = \|T\|$
Indeed

$$\|\tilde{T}x\| = \lim_{n\to\infty}\|Tx_n\| \leq \|T\|\cdot\lim_{n\to\infty}\|x\|_X = \|T\|\cdot\|x\|,$$

so $\|\tilde{T}\| \leq \|T\|$.
But $\tilde{T}$ is an extension (take $x_n = x$ if $x \in D$), so

$$\|\tilde{T}\| = \|T\|.$$

□

13.5.2. The definition of $\mathscr{F}_2$

We apply the above result to

$$\begin{array}{ll} X = \mathscr{S}, & \|\cdot\|_X = \|\cdot\|_{L^2} \\ Y = L^2_{\mathbb{C}}(\mathbb{R}), & \|\cdot\|_Y = \|\cdot\|_{L^2} \\ T = \mathscr{F} \text{ with } \|\mathscr{F}\| = 1 & \text{since } \|\hat{f}\|_{L^2} = \|f\|_{L^2}. \end{array}$$

Then we get

Proposition 416.
There exists exactly one extension of $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ to a mapping

$$\mathscr{F}_{L^2} : L^2_{\mathbb{C}}(\mathbb{R}) \to L^2_{\mathbb{C}}(\mathbb{R}), \quad f(t) \mapsto \hat{f}(\lambda) := \mathscr{F}_{L^2}[f(t)](\lambda).$$

This transformation is called the Fourier-Plancherel transform.
The Fourier-Plancherel transform has the following properties:

1. *For $f \in L^2_{\mathbb{C}}(\mathbb{R}) \cap L^1_{\mathbb{C}}(\mathbb{R})$ we have*

$$\mathscr{F}_{L^2}[f(t)](\lambda) = \int_{\mathbb{R}} f(t) \cdot e^{-2\pi i \lambda t} \, dt$$

2. *Moreover, for all $f \in L^2_{\mathbb{C}}(\mathbb{R})$ we have*

$$\mathscr{F}_{L^2}[f(t)](\lambda) = \mathrm{l.\,i.\,m.}_{R\to\infty} \int_{[-R,R]} f(t) \cdot e^{-2\pi i \lambda t} \, dt,$$

   where $\mathrm{l.\,i.\,m.}$ *stands for the limit in the L^2-norm. Thus*

$$\lim_{R\to\infty} \|\mathscr{F}_{L^2}[f(t)](\lambda) - \int_{[-R,R]} f(t) \cdot e^{-2\pi i \lambda t} \, dt\|_{L^2} = 0.$$

Moreover, the Fourier-Plancherel transform is norm-preserving:

- $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$, *for all $f \in L^2_{\mathbb{C}}(\mathbb{R})$;*
- $\langle f(t) \mid g(t)\rangle_{L^2} = \left\langle \hat{f} \mid \hat{g} \right\rangle_{L^2}$, *for all $f, g \in L^2_{\mathbb{C}}(\mathbb{R})$.*

Thus, there exists an inverse Fourier-Plancherel transform:

$$\mathscr{F}^{-1}_{L^2} : L^2_{\mathbb{C}}(\mathbb{R}) \to L^2_{\mathbb{C}}(\mathbb{R}).$$

Part V

Distributions and tempered distributions

14

Distributions

14.1. The space of test functions

Definition 417.
We collect in the set $\mathscr{D}(\mathbb{R})$ all the so-called *test functions* φ. Thereby the function

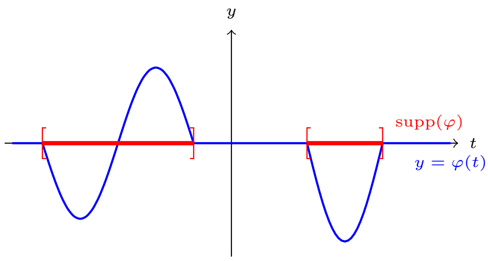
$$\varphi : \mathbb{R} \to \mathbb{C}, \qquad t \mapsto \varphi(t)$$

is a test function if

1. φ is of class C^∞ and
2. φ vanishes outside a bounded intervall (that depends on φ).

Remark 418. *If $\varphi : \mathbb{R} \to \mathbb{C}$ is a continuous function, we define the support of φ as the following closed set:*

$$\operatorname{supp}(\varphi) := \overline{\{t \in \mathbb{R} \; : \; \varphi(t) \neq 0\}}.$$

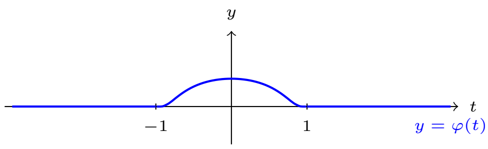


Then

$$\mathscr{D}(\mathbb{R}) = \{\varphi : \mathbb{R} \to \mathbb{C} \; : \; \varphi \text{ is of class } C^\infty \text{ with compact support}\}.$$

Example 419.
The set $\mathscr{D}(\mathbb{R})$ is quite large, even if it is not so easy to give explicitly functions that are test functions, since analytic functions cannot be test functions. Let us mention a simple test function:

$$\varphi(t) := \begin{cases} e^{-\frac{1}{1-x^2}} & \text{, if } |t| \leq 1 \\ 0 & \text{, elsewhere.} \end{cases}$$



The space $\mathscr{D}(\mathbb{R})$ has a natural structure of a linear space over $\mathbb{C}$ if one introduces the following point-wise operations:

- the addition:

$$+ : \mathscr{D}(\mathbb{R}) \times \mathscr{D}(\mathbb{R}) \to \mathscr{D}(\mathbb{R}), \qquad (\varphi, \psi) \mapsto (\varphi + \psi)(t) := \varphi(t) + \psi(t).$$

- the multiplication by a scalar:

$$\cdot : \mathbb{C} \times \mathscr{D}(\mathbb{R}) \to \mathscr{D}(\mathbb{R}), \qquad (\alpha, \varphi) \mapsto (\alpha \cdot \varphi)(t) := \alpha \cdot \varphi(t).$$

A topology on the linear space $(\mathscr{D}(\mathbb{R}), +, \cdot)$

Definition 420.
Given: a sequence $\{\varphi_n\}_{n=1}^{+\infty}$ of test functions in the linear space $(\mathscr{D}(\mathbb{R}), +, \cdot)$
we say: the sequence $\{\varphi_n\}_{n=1}^{+\infty}$ converges to φ in $\mathscr{D}$ iff:

1. there is a compact intervall $[a, b]$ such that

$$\operatorname{supp}(\varphi_n) \subset [a, b], \quad \forall n \qquad \text{and} \qquad \operatorname{supp}(\varphi) \subset [a, b].$$

2. Moreover, for $m = 0, 1, 2, \ldots$,

$$\|\varphi_n^{(m)} - \varphi^{(m)}\|_\infty := \sup_{t \in [a.b]} |\varphi_n^{(m)}(t) - \varphi^{(m)}(t)| \to 0 \qquad \text{as } n \to \infty.$$

This means that all derivatives $\varphi_n^{(m)}$ (for $m = 0, 1, 2, \ldots$) converge uniformly to $\varphi^{(m)}$.

We denote such a convergence by

$$\varphi_n \to \varphi \quad \text{in } \mathscr{D}.$$

Remark 421. *Let us remark that it is "difficult" for a given sequence $\{\varphi_n\}_{n=1}^{+\infty}$ to converge in the above sense, since there are a lots of constraints to be fulfilled.*

Proposition 422.
Equipped with this topology, the space $(\mathscr{D}(\mathbb{R}), +, \cdot)$ is a topological vector space. This means that

$$\left.\begin{array}{lll} \varphi_n \to \varphi & \text{in } \mathscr{D} \\ \psi_n \to \psi & \text{in } \mathscr{D} \\ \alpha_n \to \alpha & \text{in } \mathbb{C} \end{array}\right\} \Longrightarrow \alpha_n \cdot \varphi_n + \psi_n \to \alpha \cdot \varphi + \psi \quad \text{in } \mathscr{D}$$

14.2. Definition of a distribution

Definition 423.
Given: the space of test functions $\mathscr{D}(\mathbb{R})$
we define: *a distribution* as:
a mapping

$$T : \mathscr{D}(\mathbb{R}) \to \mathbb{C}, \quad \varphi \mapsto T\varphi = \langle T, \varphi \rangle$$

that is

1. **linear:**

$$\langle T, \alpha{\cdot}\varphi{+}\psi \rangle = \alpha{\cdot}\langle T, \varphi \rangle{+}\langle T, \psi \rangle, \qquad \forall \alpha \in \mathbb{C}, \quad \forall \varphi, \psi \in \mathscr{D}(\mathbb{R}).$$

2. **continuous at** 0**:**

$$\varphi_n \to 0 \quad \text{in } \mathscr{D} \Longrightarrow \langle T, \varphi_n \rangle \to 0 \quad \text{in } \mathbb{C}.$$

Remark 424. *In the above definition, we ask only for "continuity at* 0*". This however implies, that distributions are continuous everywhere:*

$$\varphi_n \to \varphi \quad \textit{in } \mathscr{D} \Longrightarrow \langle T, \varphi_n \rangle \to \langle T, \varphi \rangle \quad \textit{in } \mathbb{C}.$$

Definition 425.
We collect all distributions in a set we call $\mathscr{D}'(\mathbb{R})$:

$$\mathscr{D}'(\mathbb{R}) := \{ T : \mathscr{D}(\mathbb{R}) \to \mathbb{C} \ : \ T \text{ is linear and continuous} \}.$$

This is a dual space.

14.3. Locally integrable functions as distributions

Definition 426.

Given: a measurable function

$$f : \mathbb{R} \to \mathbb{C}, \quad t \mapsto f(t)$$

we say: f is locally integrable iff:

$$\exists \quad \int_K f(t)\, dt, \qquad \forall \text{ compact subset } K \subset \mathbb{R},$$

where the integral is a Lebesgue integral.
This means that the function f is integrable over any compact subset.

Definition 427.
We collect all locally integrable functions in a space we call $L^1_{\text{loc},\mathbb{C}}(\mathbb{R})$.

Remark that $L^1_{\text{loc},\mathbb{C}}(\mathbb{R})$ has a natural structure of a linear space when equipped with point-wise operations.

$$(f+g)(t) := f(t) + g(t) \qquad \text{and} \qquad (\alpha \cdot f)(t) = \alpha \cdot f(t).$$

Proposition 428.

1. *If the function $f : \mathbb{R} \to \mathbb{C}$ is continuous, then $f \in L^1_{loc,\mathbb{C}}(\mathbb{R})$.*
2. *Moreover*
$$L^p_{\mathbb{C}}(\mathbb{R}) \subset L^1_{loc,\mathbb{C}}(\mathbb{R}).$$

Proof. The last point follows from

$$\int_K f(t)\, dt = \int_K \underbrace{1}_{\in L^q} \cdot \underbrace{f(t)}_{\in L^p}\, dt < +\infty$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

□

Example 429.
We have
$$f(t) := e^{t^2} \in L^1_{\mathrm{loc},\mathbb{C}}(\mathbb{R}).$$

Proposition 430.
Any function $f \in L^1_{loc,\mathbb{C}}(\mathbb{R})$ can be viewed as a distribution T_f via
$$\langle T_f, \varphi \rangle := \int_{\mathbb{R}} f(t) \cdot \varphi(t)\, dt, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}).$$

Proof. **(I)** T_f **is well-defined** since
$$\int_{\mathbb{R}} f(t) \cdot \varphi(t)\, dt = \int_{\mathrm{supp}(\varphi)} \underbrace{f(t)}_{\in L^1} \cdot \underbrace{\varphi(t)}_{\in L^\infty}\, dt,$$
i.e. the integrand $f(t) \cdot \varphi(t)$ belongs to L^1.
(II) T_f **is linear:**
This follows from
$$\int_{\mathbb{R}} f(t) \left(\alpha \cdot \varphi(t) + \psi(t)\right)\, dt = \alpha \cdot \int_{\mathbb{R}} f(t)\varphi(t)\, dt + \int_{\mathbb{R}} f(t)\psi(t)\, dt.$$
(III) T_f **is continuous at** 0**:**
Indeed, if the sequence $\{\varphi_n\}_{n=1}^{+\infty}$ converges in $\mathscr{D}$ to 0, and if we denote by $[a,b]$ a compact intervall containing all $\mathrm{supp}(\varphi_n)$, we have
$$\begin{aligned} |\langle T_f, \varphi_n \rangle| &= \left| \int_{[a,b]} f(t) \cdot \varphi(t)\, dt \right| \leq \int_{[a,b]} |f(t)| \cdot |\varphi(t)|\, dt \\ &\leq \underbrace{\sup_{t \in [a,b]} |\varphi_n(t)|}_{\to 0} \cdot \underbrace{\int_{[a,b]} |f(t)|\, dt}_{<+\infty}, \end{aligned}$$
so that
$$\lim_{n \to \infty} \langle T_f, \varphi_n \rangle = 0.$$
□

Definition 431.

Given: a distribution T

we say: T is null (is zero or vanishes) on an open set $\Omega \subset \mathbb{R}$ iff:

$$\langle T, \varphi \rangle = 0, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}) \text{ with } \operatorname{supp}(\varphi) \subset \Omega.$$

Definition 432.

The support $\operatorname{supp}(T)$ of a distribution T is the complement of the largest open set on which T is null.

Proposition 433.

For any locally integrable function f, we have

$$\operatorname{supp}(T) = \operatorname{supp}(f).$$

Thereby $\operatorname{supp}(f)$ *is defined as*

$$\complement O, \qquad O = \bigcup_\iota O_\iota,$$

where O_ι is any open set with $f|_{O_\iota} = 0$.

Identification

Proposition 434.

1. *If $f \in L^1_{loc,\mathbb{C}}(\mathbb{R})$ is such that*

$$\langle T_f, \varphi \rangle = 0, \quad \forall \varphi \in \mathscr{D}(\mathbb{R}),$$

then

$$f = 0 \text{ a.e.}$$

2. *If $f, g \in L^1_{loc,\mathbb{C}}(\mathbb{R})$ are such that*

$$\langle T_f, \varphi \rangle = \langle T_g, \varphi \rangle, \quad \forall \varphi \in \mathscr{D}(\mathbb{R}),$$

then

$$f = g \text{ a.e.}$$

We may thus identify $L^1_{\text{loc},\mathbb{C}}(\mathbb{R})$ as a subspace of $\mathscr{D}'$.

Definition 435.
Distributions of the form T_f with $f \in L^1_{\text{loc},\mathbb{C}}(\mathbb{R})$ are called regular distributions.

Remark 436. *Remark that there exists nonregular distributions. Thus*

$$L^1_{loc,\mathbb{C}}(\mathbb{R}) \subsetneq \mathscr{D}'.$$

The following example gives such a nonregular distribution.

Dirac distribution

Example 437.
Consider the mapping

$$\delta : \mathscr{D}(\mathbb{R}) \to \mathbb{C}, \qquad \langle \delta, \varphi \rangle := \varphi(0).$$

(I) δ is a distribution:
Indeed

1. δ is well-defined for all $\varphi \in \mathscr{D}(\mathbb{R})$.
2. δ is linear since

$$(\alpha \cdot \varphi + \psi)(0) = \alpha \cdot \varphi(0) + \psi(0), \qquad \forall \alpha \in \mathbb{C}, \quad \forall \varphi, \psi \in \mathscr{D}(\mathbb{R}).$$

3. δ is continuous at 0, since

$$\varphi_n \to 0 \quad \text{in } \mathscr{D} \Longrightarrow \lim_{n\to\infty} \varphi_n(0) = 0 \Longrightarrow \lim_{n\to\infty} \langle \delta, \varphi_n \rangle = 0.$$

(II) δ is not a regular distribution:
Indeed, remark that

$$\text{supp}(\delta) = \{0\}.$$

Thus, if there would exist some $f \in L^1_{\text{loc},\mathbb{C}}(\mathbb{R})$ with $\delta = T_f$, we would have

$$\text{supp}(f) = \{0\},$$

i.e. $f = 0$. But this would mean that

$$\langle T_f, \varphi \rangle = 0, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}),$$

and this contradicts

$$\langle \delta, \varphi \rangle = \varphi(0)$$

as soon as one chooses an element $\varphi \in \mathscr{D}(\mathbb{R})$ with $\varphi(0) \neq 0$.

14.4. Elementary operations on distributions

14.4.1. Translate of a distribution

For any $f \in L^1_{\text{loc},\mathbb{C}}(\mathbb{R})$ and for any $a \in \mathbb{R}$, we put

$$(\tau_a f)(t) := f(t-a), \qquad \forall t \in \mathbb{R}.$$

Remark that, $\forall \varphi \in \mathscr{D}(\mathbb{R})$, we have

$$\begin{aligned} \langle T_{\tau_a f}, \varphi \rangle &= \int_{\mathbb{R}} (\tau_a f)(t) \cdot \varphi(t)\, dt = \int_{\mathbb{R}} f(t-a) \cdot \varphi(t)\, dt \\ &= \int_{\mathbb{R}} f(t) \cdot \varphi(t+a)\, dt \\ &= \langle T_f, \tau_{-a} \varphi \rangle \end{aligned}$$

This motivates the following definition.

The translate of a distribution

Definition 438.

Given: a distribution T and a constant $a > 0$

we define: the translate $\tau_a T$ as:
the distribution given by

$$\langle \tau_a T, \varphi \rangle = \langle T, \tau_{-a} \varphi \rangle, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}).$$

Periodic distributions

Definition 439.
Given: a distribution T and a constant $a > 0$
we say: T is a-periodic iff:

$$\tau_a T = T,$$

Remark 440. *Thus T is a-periodic if*

$$\langle \tau_a T, \varphi \rangle = \langle T, \varphi \rangle, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}),$$

i.e. if

$$\langle T, \tau_{-a} \varphi \rangle = \langle T, \varphi \rangle, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}).$$

Example 441.
As an example of a 2π-periodic distribution, we cite

$$T_{\cos t}$$

with

$$\langle T_{\cos t}, \varphi \rangle = \int_{\mathbb{R}} \cos t \cdot \varphi(t)\, dt, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}).$$

14.4.2. The product of a function and a distribution

For any $f \in L^1_{\text{loc},\mathbb{C}}(\mathbb{R})$ and for any g of class C^∞, we have

$$\begin{aligned} \langle T_{f \cdot g}, \varphi \rangle &= \int_{\mathbb{R}} (f(t) \cdot g(t)) \cdot \varphi(t)\, dt \\ &= \int_{\mathbb{R}} f(t) \cdot (g(t) \cdot \varphi(t))\, dt \\ &= \langle T_f, g \cdot \varphi \rangle, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}). \end{aligned}$$

Remark thereby that

- $f \cdot g \in L^1_{\text{loc},\mathbb{C}}(\mathbb{R})$ if g is of class C^∞;
- $f \cdot \varphi \in \mathscr{D}(\mathbb{R})$ if g is of class C^∞.

This motivates the following definition.

product of a smooth function and of a distribution

Definition 442.
Given: a smooth function $f \in C^\infty_{\mathbb{C}}(\mathbb{R})$ and of a distribution T
we define: the product of f and T as:

$$\langle g \cdot T, \varphi \rangle = \langle T, g \cdot \varphi \rangle, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}).$$

Example 443.
For any g of class C^∞, we have

$$\langle g \cdot \delta, \varphi \rangle = \langle \delta, g \cdot \varphi \rangle = g(0) \cdot \varphi(0)$$

if δ is the Dirac distribution

$$\langle \delta, \varphi \rangle = \varphi(0), \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}).$$

Thus

$$g \cdot \delta = g(0) \cdot \delta.$$

14.4.3. The derivative of a distribution

For any $f \in L^1_{\text{loc},\mathbb{C}}(\mathbb{R})$ that has a derivative f' and for any $\varphi \in \mathscr{D}(\mathbb{R})$ whose support is contained in a compact intervall $[a,b]$, we have

$$\begin{aligned}
\langle T_{f'}, \varphi \rangle &= \int_{\mathbb{R}} f'(t) \cdot \varphi(t)\, dt \\
&= \int_{[a,b]} f'(t) \cdot \varphi(t)\, dt \\
&= \underbrace{f(t) \cdot \varphi(t)|_a^b}_{=0} - \int_{[a,b]} f(t) \cdot \varphi'(t)\, dt \\
&= -\int_{\mathbb{R}} f(t) \cdot \varphi'(t)\, dt \\
&= -\langle T_f, \varphi' \rangle.
\end{aligned}$$

This motivates the following definition.

Every distribution T has derivatives (of any order)

Definition 444.
Given: a distribution T
we define: its derivatives (of any order) as:

1. $\langle T', \varphi \rangle = -\langle T, \varphi' \rangle, \quad \forall \varphi \in \mathscr{D}(\mathbb{R});$
2. $\langle T'', \varphi \rangle = \langle T, \varphi'' \rangle, \quad \forall \varphi \in \mathscr{D}(\mathbb{R});$
3. $\langle T''', \varphi \rangle = -\langle T, \varphi''' \rangle, \quad \forall \varphi \in \mathscr{D}(\mathbb{R});$
4. $\ldots.$

Hence, for $k = 1, 2, 3, \ldots$,

$$\boxed{\langle T^{(k)}, \varphi \rangle = (-1)^k \langle T, \varphi^{(k)} \rangle, \quad \forall \varphi \in \mathscr{D}(\mathbb{R})}$$

Remark 445. *Let us insist on the fact that, even if a function $f \in L^1_{loc,\mathbb{C}}(\mathbb{R})$ has no derivative (or no derivative in some points), this function has a distributional derivative.*

Example 446.
Consider the Heaviside function

$$u : \mathbb{R} \to \mathbb{C}, \qquad t \mapsto u(t) := \begin{cases} 1 & \text{, if } x \geq 0 \\ 0 & \text{, elsewhere} \end{cases}$$

Then:

- We have

$$\langle T_u, \varphi \rangle = \int_0^{+\infty} 1 \cdot \varphi(t)\, dt = \int_0^{+\infty} \varphi(t)\, dt, \quad \forall \varphi \in \mathscr{D}(\mathbb{R})$$

- u has no derivative at $t = 0$. In the distributional sense, T_u has a derivative T_u' defined by

$$\begin{aligned} \langle T_u', \varphi \rangle &= -\langle T_u, \varphi' \rangle = -\int_0^{+\infty} \varphi'(t)\, dt = -\int_0^{b} \varphi'(t)\, dt \\ &= -\varphi(b) + \varphi(0) = \varphi(0) = \langle \delta, \varphi \rangle. \end{aligned}$$

We have thereby chosen $b > 0$ in such a way that $\text{supp}(\varphi) \subset]-\infty, b]$. Thus

$$\boxed{T_u' = \delta \qquad (\text{in } \mathscr{D}').}$$

Example 447.
The Dirac distribution

$$\langle \delta, \varphi \rangle = \varphi(0), \quad \forall \varphi \in \mathscr{D}(\mathbb{R})$$

has a derivative defined by

$$\langle \delta', \varphi \rangle \;=\; -\langle \delta, \varphi' \rangle = -\varphi'(0), \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}).$$

Thus, the second derivative of the Heaviside is given by

$$\langle T_u'', \varphi \rangle = \langle T_u, \varphi'' \rangle = \int_0^{+\infty} \varphi''(t)\, dt = -\varphi'(0), \quad \forall \varphi \in \mathscr{D}(\mathbb{R}).$$

This may be written as

$$\boxed{T_u' = \delta, \qquad T_u'' = \delta' \qquad (\text{in } \mathscr{D}').}$$

Let us consider a function

$$f : \mathbb{R} \to \mathbb{C}, \quad t \mapsto f(t)$$

that is continuous, except in a finite number of points

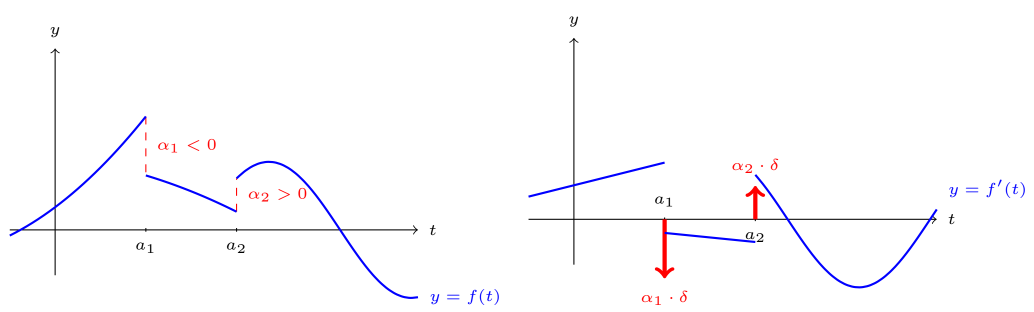
$$a_1 < a_2 < \cdots < a_n.$$

Suppose that in each point of discontinuity a_k $(k = 1, 2, \ldots, n)$, the function f has a simple jump; this means that the limits

$$f(a_k^+) := \lim_{t \to a_k^+} f(t) \qquad \text{and} \qquad f(a_k^-) := \lim_{t \to a_k^-} f(t)$$

both exist (in $\mathbb{R}$) and that

$$\alpha_k := f(a_k^+) - f(a_k^-) \neq 0.$$



Then, using partial integration on each intervall where the derviative exists, we get

$$
\begin{aligned}
\langle T_f', \varphi \rangle &= -\langle T_f, \varphi' \rangle \\
&= -\int_{-\infty}^{a_1} f(t)\cdot\varphi'(t)\,dt - \\
&\quad -\sum_{k=1}^{n-1}\int_{a_k}^{a_{k+1}} f(t)\cdot\varphi'(t)\,dt - \int_{a_n}^{+\infty} f(t)\cdot\varphi'(t)\,dt \\
&= -\,f(t)\cdot\varphi(t)\big|_{-\infty}^{a_1^-} + \int_{-\infty}^{a_1} f'(t)\cdot\varphi(t)\,dt - \\
&\quad -\sum_{k=1}^{n-1}\left(f(t)\cdot\varphi(t)\big|_{a_k^+}^{a_{k+1}^-} - \int_{a_k}^{a_{k+1}} f'(t)\cdot\varphi(t)\,dt \right) - \\
&\quad -\,f(t)\cdot\varphi(t)\big|_{a_n^+}^{+\infty} + \int_{a_n}^{+\infty} f'(t)\cdot\varphi(t)\,dt
\end{aligned}
$$

Remark that

- $f(t)\cdot\varphi(t)\big|_{-\infty}^{a_1^-} = f(a_1^-)\varphi(a_1)$;
- $f(t)\cdot\varphi(t)\big|_{a_k^+}^{a_{k+1}^-} = f(a_{k+1}^-)\varphi(a_{k+1}) - f(a_k^+)\varphi(a_k)$, so

$$
\begin{aligned}
\sum_{k=1}^{n-1} f(t)\cdot\varphi(t)\big|_{a_k^+}^{a_{k+1}^-} &= \sum_{k=1}^{n-1} f(a_{k+1}^-)\varphi(a_{k+1}) - \sum_{k=1}^{n-1} f(a_k^+)\varphi(a_k) \\
&= \sum_{k=2}^{n} f(a_k^-)\varphi(a_k) - \sum_{k=1}^{n-1} f(a_k^+)\varphi(a_k) \\
&= f(a_n^-)\varphi(a_n) + \sum_{k=2}^{n-1} \alpha_k\cdot\varphi(a_k) - f(a_1^+)\varphi(a_1)
\end{aligned}
$$

- $f(t)\cdot\varphi(t)\big|_{a_n^+}^{+\infty} = -f(a_n^+)\varphi(a_n)$.

Thus we get

$$\begin{aligned}\langle T_f', \varphi \rangle &= -f(t)\cdot\varphi(t)\big|_{-\infty}^{a_1^-} + \int_{-\infty}^{a_1} f'(t)\cdot\varphi(t)\,dt \\ &\quad -\sum_{k=1}^{n-1}\left(f(t)\cdot\varphi(t)\big|_{a_k^+}^{a_{k+1}^-} - \int_{a_k}^{a_{k+1}} f'(t)\cdot\varphi(t)\,dt\right) \\ &\quad - f(t)\cdot\varphi(t)\big|_{a_n^+}^{+\infty} + \int_{a_n}^{+\infty} f'(t)\cdot\varphi(t)\,dt \\ &= \sum_{k=1}^{n}\alpha_k\cdot\varphi(a_k) + \int_{-\infty}^{a_1} f'(t)\cdot\varphi(t)\,dt + \\ &\quad +\sum_{k=1}^{n-1}\int_{a_k}^{a_{k+1}} f'(t)\cdot\varphi(t)\,dt + \int_{a_n}^{+\infty} f'(t)\cdot\varphi(t)\,dt \\ &= \sum_{k=1}^{n}\alpha_k\cdot\varphi(a_k) + \int_{\mathbb{R}} f'(t)\cdot\varphi(t)\,dt.\end{aligned}$$

Let us introduce the notation (for all $a \in \mathbb{R}$)

$$\delta_a := \tau_a\delta$$

i.e.

$$\langle \delta_a, \varphi\rangle = \langle \delta, \tau_{-a}\varphi\rangle = \varphi(t+a)\big|_{t=0} = \varphi(a)$$

i.e.

$$\langle \delta_a, \varphi\rangle := \varphi(a), \qquad \forall \varphi \in \mathscr{D}(\mathbb{R})$$

Clearly

$$\delta_0 = \delta.$$

With this notations, we get now

$$\begin{aligned}\langle T_f', \varphi\rangle &= \sum_{k=1}^{n}\alpha_k\cdot\varphi(a_k) + \int_{\mathbb{R}} f'(t)\cdot\varphi(t)\,dt \\ &= \sum_{k=1}^{n}\alpha_k\cdot\langle\delta_{a_k},\varphi\rangle + \langle f', \varphi\rangle \\ &= \left\langle \sum_{k=1}^{n}\alpha_k\cdot\delta_{a_k} + f', \varphi\right\rangle \qquad \forall\varphi\in\mathscr{D}(\mathbb{R}).\end{aligned}$$

We write this as

$$\boxed{T_f' = \sum_{k=1}^{n}\alpha_k\cdot\delta_{a_k} + T_{f'}.}$$

Proposition 448.

<u>Hyp</u> *Let us consider a function*

$$f : \mathbb{R} \to \mathbb{C}, \quad t \mapsto f(t)$$

that is continuous, except in a finite number of points

$$a_1 < a_2 < \cdots < a_n.$$

Suppose that in each point of discontinuity a_k ($k = 1, 2, \ldots, n$), the function f has a simple jump; this means that the limits

$$f(a_k^+) := \lim_{t \to a_k^+} f(t) \qquad \text{and} \qquad f(a_k^-) := \lim_{t \to a_k^-} f(t)$$

both exist (in $\mathbb{R}$) and that

$$\alpha_k := f(a_k^+) - f(a_k^-) \neq 0.$$

<u>Concl</u> *Then*

$$\boxed{T_f' = \sum_{k=1}^{n} \alpha_k \cdot \delta_{a_k} + T_{f'}.}$$

Remark 449. *In the above proposition, we may admit functions having an infinitely countable number of simple discontinuities, if the points of discontinuity have no point of accumulation.*

Indeed, in this case, only a finite number of points of discontinuity are in the compact set $\operatorname{supp}(\varphi)$, $\forall \varphi \in \mathscr{D}(\mathbb{R})$.

Thus we get the following extension:

Proposition 450.

Hyp *Let us consider a function*

$$f : \mathbb{R} \to \mathbb{C}, \quad t \mapsto f(t)$$

that is continuous, except in a countably infinite number of points

$$\{a_n\}_{n=1}^{+\infty}.$$

Let us assume that these points of discontinuity have no point of accumulation.

Suppose further that in each point of discontinuity a_k ($k = 1, 2, \ldots$), the function f has a simple jump; this means that the limits

$$f(a_k^+) := \lim_{t \to a_k^+} f(t) \qquad \text{and} \qquad f(a_k^-) := \lim_{t \to a_k^-} f(t)$$

both exist (in $\mathbb{R}$) and that

$$\alpha_k := f(a_k^+) - f(a_k^-) \neq 0.$$

Concl *Then*

$$\boxed{T_f' = \sum_{k=1}^{\infty} \alpha_k \cdot \delta_{a_k} + T_{f'}.}$$

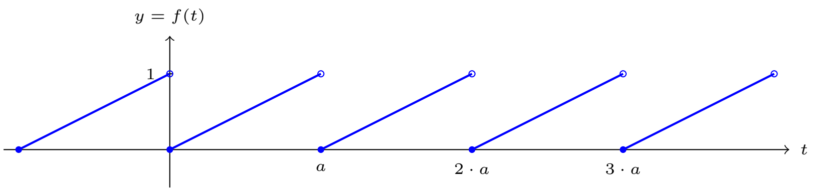
Remark that the sum

$$\sum_{k=1}^{\infty} \alpha_k \cdot \langle \delta_{a_k}, \varphi \rangle$$

is a finite sum for each $\varphi \in \mathscr{D}(\mathbb{R})$.

Example 451.
Consider the a-periodic function f given by

$$f(t) = t/a, \qquad \text{for } t \in [0, a[$$



Then

$$T'_f = \frac{1}{a} - \sum_{k\in\mathbb{Z}} \delta_{k\cdot a}$$

i.e.

$$\langle T'_f, \varphi\rangle = \int_{\mathbb{R}} \varphi(t)\; dt - \sum_{k\in\mathbb{Z}} \varphi(k\cdot a).$$

Remark that the above integral is an integral over the compact set $\text{supp}(\varphi)$ and that the above sum in in fact a finite sum.

14.4.4. Convergence of distributions

Definition 452.

Given: a sequence of distributions $\{T_n\}_{n=1}^{+\infty}$ and another distribution T

we say: this sequence of distributions $\{T_n\}_{n=1}^{+\infty}$ converges to T iff:

$$\lim_{n\to\infty} \langle T_n, \varphi\rangle = \langle T,\varphi\rangle, \qquad \varphi \in \mathscr{D}(\mathbb{R}).$$

This notion of convergence is noted as

$$\boxed{T_n \to T \qquad \text{(in } \mathscr{D}'\text{)}.}$$

Example 453.
Consider a sequence of real numbers $\{a_n\}_{n=1}^{+\infty}$ converging to a. Then

$$\delta_{a_n} \to \delta_a \qquad \text{(in } \mathscr{D}'\text{)}$$

since, $\forall\varphi \in \mathscr{D}(\mathbb{R})$, we have

$$\lim_{n\to\infty} \langle \delta_{a_n}, \varphi\rangle = \lim_{n\to\infty} \varphi(a_n) = \varphi(a) = \langle \delta_a, \varphi\rangle.$$

We have used the fact that all test functions are continuous.

We refer to the following property as the *continuity of the derivation.*

Proposition 454.

<u>Hyp</u> *The sequence of test functions* $\{T_n\}_{n=1}^{+\infty}$ *converges to the distribution* T*, i.e. if*

$$T_n \to T \qquad (\text{in } \mathscr{D}').$$

<u>Concl</u> *The sequence of test functions* $\{T_n'\}_{n=1}^{+\infty}$ *converges to* T'*, i.e.*

$$T_n' \to T' \qquad (\text{in } \mathscr{D}').$$

Proof. Indeed, $\forall \varphi \in \mathscr{D}(\mathbb{R})$, we have

$$\lim_{n\to\infty} \langle T_n', \varphi\rangle \quad = \quad -\lim_{n\to\infty} \langle T_n, \varphi'\rangle = -\langle T, \varphi'\rangle = \langle T', \varphi\rangle.$$

This gives the claim! □

Example 455.
Consider a sequence of real numbers $\{\lambda_n\}_{n=1}^{+\infty}$ with

$$\lim_{n\to\infty} \lambda_n = +\infty.$$

Then, by the Riemann-Lesgegue Lemma, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(t) \cdot e^{-2\pi i \lambda_n t}\, dt = \lim_{n\to\infty} \hat{\varphi}(\lambda_n) = 0,$$

if $\hat{\varphi}$ is the Fourier transformed of a test function φ.
Thus

$$\lim_{n\to\infty} \langle T_{e^{-2\pi i \lambda_n t}}, \varphi\rangle = 0, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}),$$

i.e.

$$T_{e^{-2\pi i \lambda_n t}} \to 0 \qquad (\text{in } \mathscr{D}')$$

despite the fact that the sequence of functions $\{e^{-2\pi i \lambda_n t}\}$ does not converge!

Proposition 456.

<u>Hyp</u> *The sequence* $\{f_n\}_{n=1}^{+\infty}$ *of functions in* $L^2_{\mathbb{C}}(\mathbb{R})$ *converges to a function* f *with respect to the* L^2*-norm.*

Concl *Then*

$$T_{f_n} \to T_f \qquad \text{(in } \mathscr{D}'\text{)}.$$

Proof. This follows from the fact that, $\forall \varphi \in \mathscr{D}(\mathbb{R})$, we have

$$\begin{aligned} |\langle T_{f_n}, \varphi\rangle - \langle T_f, \varphi\rangle| &= \left| \int_{\mathbb{R}} (f_n(t) - f(t)) \cdot \varphi(t)\, dt \right| \\ &\leq \int_{\mathbb{R}} |f_n(t) - f(t)| \cdot |\varphi(t)|\, dt \\ &\leq \|f_n - f\|_{L^2} \cdot \|\varphi\|_{L^2} \to 0. \end{aligned}$$

□

Remark that the integral in the above proof is in fact an integral over the compact set $\text{supp}(\varphi)$. Thus, a similar proof works for periodic signals.

Proposition 457.

Hyp *The sequence* $\{f_n\}_{n=1}^{+\infty}$ *of* a*-periodic functions (with* $a > 0$*) in* $L^2_{\mathbb{C}}(\mathbb{R})$ *converges to an* a*-periodic function* f *in the sense that*

$$\lim_{n\to\infty} \int_0^a |f_n(t) - f(t)|^2\, dt = 0.$$

Concl *Then*

$$T_{f_n} \to T_f \qquad \text{(in } \mathscr{D}'\text{)}.$$

Proposition 458.

Hyp *The sequence* $\{f_n\}_{n=1}^{+\infty}$ *of measurable functions converges a.e. to a function* f *and if there exists a majoration* $g \in L^1(\mathbb{R})$ *with*

$$|f_n(t)| \leq g(t) \quad \text{a.e. on } \mathbb{R}.$$

<u>*Concl*</u> *Then*

$$T_{f_n} \to T_f \qquad (\text{in } \mathscr{D}').$$

Proof. By Lebesgue's dominated convergence theorem, we have, $\forall \varphi \in \mathscr{D}(\mathbb{R})$,

$$\begin{aligned}
\lim_{n\to\infty} \langle T_{f_n}, \varphi\rangle &= \lim_{n\to\infty} \int_{\mathbb{R}} f_n(t)\cdot\varphi(t)\, dt \\
&= \int_{\mathbb{R}} \lim_{n\to\infty} f_n(t)\cdot\varphi(t)\, dt = \int_{\mathbb{R}} f(t)\cdot\varphi(t)\, dt \\
&= \langle T_f, \varphi\rangle.
\end{aligned}$$

This gives the claim. □

14.4.5. The Dirac comb

Definition 459.

<u>Given:</u> a fixed $a > 0$

we define: <u>the Dirac comb Ш_a</u> as:

the distribution

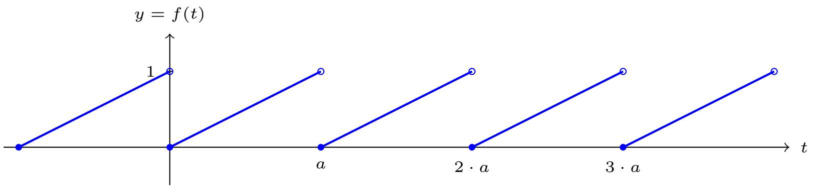
$$\text{Ш}_a := \sum_{k\in\mathbb{Z}} \delta_{k\cdot a}.$$

(Ш is pronounced as "shah").

Remark 460. *The Dirac comb is an a-periodic distribution!*

Consider again the a-periodic signal defined by

$$f(t) = t/a, \qquad \text{for } t \in [0, a[$$



We yet now that

$$T_f' = \frac{1}{a} - \sum_{k\in\mathbb{Z}} \delta_{k\cdot a} = \frac{1}{a} - \text{Ш}_a.$$

Since $f \in L^2_{\mathbb{C}}([0, a])$, we know that

$$f(t) = \frac{1}{2} + \frac{i}{2\pi} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{n} \cdot e^{2\pi i \frac{k}{a} t} \qquad \text{in } L^2_{\mathbb{C}}([0, a]).$$

This is the Fourier series expansion of f, an expansion that converges in the L^2-norm. Thus, we have convergence in the sense of distributions:

$$T_f = \frac{1}{2} + \lim_{n\to\infty} \frac{i}{2\pi} \sum_{\substack{k=-n \\ k\neq 0}}^{n} \frac{1}{n} \cdot e^{2\pi i \frac{k}{a} t} \qquad \text{in } \mathscr{D}'.$$

By the continuity of the derivation, this leads us to

$$T_f' = -\lim_{n\to\infty} \frac{1}{a} \sum_{\substack{k=-n \\ k\neq 0}}^{n} e^{2\pi i \frac{k}{a} t} = \frac{1}{a} - \lim_{n\to\infty} \frac{1}{a} \sum_{k=-n}^{n} e^{2\pi i \frac{k}{a} t}$$

Thus we have got

$$T_f' = \frac{1}{a} - \lim_{n\to\infty} \frac{1}{a} \sum_{k=-n}^{n} e^{2\pi i \frac{k}{a} t} = \frac{1}{a} - \text{Ш}_a.$$

This gives us the following expression for the Dirac comb:

$$\text{Ш}_a = \sum_{k\in\mathbb{Z}} e^{2\pi i \frac{k}{a} t}$$

with

$$\text{Ш}_a = \sum_{k\in\mathbb{Z}} \delta_{k\cdot a}.$$

The relation

$$\boxed{\sum_{k\in\mathbb{Z}} \delta_{k\cdot a} = \sum_{k\in\mathbb{Z}} e^{2\pi i \frac{k}{a} t}}$$

can be interpreted as the Fourier series expansion of Ш in the sens of distributions. We will develop this later!

15

Tempered distributions

15.1. Tempered distributions

If $f \in L^1_{\mathbb{C}}(\mathbb{R})$ and $\varphi \in \mathscr{D}(\mathbb{R})$, on has

$$\begin{aligned} \langle T_{\hat{f}}, \varphi \rangle &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t) \cdot e^{-2\pi i \lambda t} \, dt \right) \varphi(\lambda) d\lambda \\ &= \int_{\mathbb{R}} f(t) \left(\int_{\mathbb{R}} \varphi(\lambda) \cdot e^{-2\pi i \lambda t} \, d\lambda \right) dt \\ &= \int_{\mathbb{R}} f(t) \cdot \hat{\varphi}(t) \, dt \end{aligned}$$

This computation would motivate the following definition of the Fourier transform of a distribution

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}).$$

Unfortunately,

$$\varphi \in \mathscr{D}(\mathbb{R}) \Longrightarrow \hat{\varphi} \notin \mathscr{D}(\mathbb{R})$$

In order to overcome this difficulty, we will apply the above definition of a Fourier transform only to so-called *tempered distributions*. Such distributions will be defined not only for $\varphi \in \mathscr{D}(\mathbb{R})$, a tempered distribution will be defined on the larges space $\mathscr{S}$. Since

$$\varphi \in \mathscr{S} \Longrightarrow \hat{\varphi} \in \mathscr{S},$$

the above definition will make sense.

Let us formalize this procedure!

15.1.1. The topology on $\mathscr{S}$

First of all, remark that, as linear spaces, we have

$$\mathscr{D}(\mathbb{R}) \subset \mathscr{S}.$$

We introduce now a notion of convergence on $\mathscr{S}$ that reduces, for sequences in $\mathscr{D}(\mathbb{R})$, to convergence in $\mathscr{D}$.

Definition 461.
Let us consider a sequence $\{\varphi_n\}_{n=1}^{+\infty}$ in $\mathscr{S}$ in and a fixed element $\varphi \in \mathscr{S}$.

1. the sequence $\{\varphi_n\}_{n=1}^{+\infty}$ converges in $\mathscr{S}$ to 0:

   for all n and $m \in \{0, 1, 2, 3, \ldots\}$, we have

$$\lim_{n\to\infty} \|t^n \cdot \varphi^{(m)}(t)\|_\infty = \sup_{t\in\mathbb{R}} |t^n \cdot \varphi^{(m)}(t)| = 0.$$

   We denote this by

$$\varphi_n \to 0 \quad \text{in } \mathscr{S}.$$

2. the sequence $\{\varphi_n\}_{n=1}^{+\infty}$ in $\mathscr{S}$ converges in $\mathscr{S}$ to φ:

$$\varphi_n - \varphi \to 0 \quad \text{in } \mathscr{S}.$$

   We denote this by

$$\varphi_n \to \varphi \quad \text{in } \mathscr{S}.$$

Remark 462. *We note that, whenever $\varphi_n \to 0$ in $\mathscr{S}$, then*

$$\lim_{n\to\infty} \|(1+t^2)^n \cdot \varphi^{(m)}(t)\|_\infty = \sup_{t\in\mathbb{R}} |(1+t^2)^n \cdot \varphi^{(m)}(t)| = 0,$$

for all n and $m \in \{0, 1, 2, 3, \ldots\}$.

Remark 463. *Let us remark that the inclusion $\mathscr{D}(\mathbb{R}) \subset \mathscr{S}$ is continuous:*

$$\varphi_n \to \varphi \quad \text{in } \mathscr{D} \Longrightarrow \varphi_n \to \varphi \quad \text{in } \mathscr{S}.$$

15.1.2. Definition of a tempered distribution

Definition 464.
A tempered distribution T is a mapping

$$T : \mathscr{S} \to \mathbb{C}, \qquad \varphi \mapsto T\varphi := \langle T, \varphi\rangle$$

that is

- **linear:** $\forall \alpha \in \mathbb{C}, \forall \varphi, \psi \in \mathscr{S}$,

$$\langle T, \alpha\cdot\varphi + \psi\rangle = \alpha \cdot \langle T, \varphi\rangle + \langle T, \psi\rangle.$$

- **continuous at 0:**

$$\varphi_n \to \varphi \quad \text{in } \mathscr{S} \Longrightarrow \lim_{n\to\infty} \langle T, \varphi_n\rangle = \langle T, \varphi\rangle.$$

Again, continuity at 0 implies continuity everywhere.

Definition 465.
We denote by $\mathscr{S}'$ the space of all tempered distributions.

Tempered distributions may be viewed as distributions

Proposition 466.

Hyp	*Consider a tempered distrubtion* $T : \mathscr{S} \to \mathbb{C}$
Concl	*The restriction* $T|_{\mathscr{D}(\mathbb{R})}$ *of* T *to the subspace* $\mathscr{D}(\mathbb{R}) \subset \mathscr{S}$ *is a distribution, i.e.*

$$T \in \mathscr{S}' \Longrightarrow T|_{\mathscr{D}(\mathbb{R})} \in \mathscr{D}'.$$

Example 467.
The Dirac distribution δ is a tempered distribution:

$$\langle \delta, \varphi \rangle = \varphi(0), \qquad \forall \varphi \in \mathscr{S}.$$

Remark 468. *Not all distributions are tempered distributions. As an example, consider the distribution*

$$T_{e^{t^2}}$$

that is not *a tempered distribution.*

15.2. Functions as tempered distributions

Definition 469.
A measurable function

$$f : \mathbb{R} \to \mathbb{C}, \quad t \mapsto f(t)$$

is slowly increasing if

$$\exists N \in \mathbb{N} \text{ such that } \qquad \sup_{t\in\mathbb{R}} \frac{|f(t)|}{(1+t^2)^N} < +\infty.$$

On usually says that slowly increasing functions have at most a polynomial growth at infinity.

Proposition 470.

Every slowly increasing function $f : \mathbb{R} \to \mathbb{C}$ can be viewed as a tempered distribution T_f through

$$\langle T_f, \varphi\rangle = \int_{\mathbb{R}} f(t) \cdot \varphi(t)\, dt.$$

Proof. First of all, T_f is well-defined. Indeed, suppose that N is such that

$$\sup_{t\in\mathbb{R}} \frac{|f(t)|}{(1+x^2)^N}.$$

Then $\forall \varphi \in \mathscr{S}$, we have

$$|f(t)\cdot\varphi(t)| \quad = \quad \underbrace{\frac{|f(t)|}{(1+t^2)^{N+1}}}_{\leq \frac{\text{const}}{(1+t^2)} \in L^1} \cdot \underbrace{(1+t^2)\cdot|\varphi(t)|}_{\in L^\infty},$$

so

$$\int_{\mathbb{R}} f(t)\cdot\varphi(t)\, dt \in \mathbb{C}.$$

That fact that T_f is linear follows immediately from the definition.

So, we arrive to the conclusion if we can show that T_f is continuous.

So let us assume that $\varphi_n \to 0$ in $\mathscr{S}$. If we choose N as at the beginning of this proof, we get

$$\begin{aligned}
|\langle T_f, \varphi_n\rangle| &= \left|\int_{\mathbb{R}} f(t)\cdot\varphi_n(t)\, dt\right| \\
&\leq \int_{\mathbb{R}} \frac{|f(t)|}{(1+t^2)^{N+1}} \cdot (1+t^2)\cdot|\varphi_n(t)|\, dt \\
&\leq \underbrace{\|(1+t^2)\cdot|\varphi_n(t)|\,\|_\infty}_{\to 0} \cdot \int_{\mathbb{R}} \underbrace{\frac{|f(t)|}{(1+t^2)^{N+1}}}_{\in L^1}\, dt,
\end{aligned}$$

i.e.

$$\lim_{n\to\infty} \langle T_f, \varphi_n\rangle = 0.$$

Thus we are done!

□

Proposition 471.
All functions $f \in L^p_{\mathbb{C}}(\mathbb{R})$ *with* $p \in [1, +\infty[$ *can be viewed as a tempered distribution* T_f *through*

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}} f(t) \cdot \varphi(t)\, dt \qquad \forall \varphi \in \mathscr{S}.$$

Proof. The proof is similar to the previous one.
Remark that one uses a relation of the form

$$\int_{\mathbb{R}} |f(t) \cdot \varphi(t)|\, dt \quad = \quad \int_{\mathbb{R}} \underbrace{|f(t)|}_{\in L^p} \cdot \underbrace{\frac{1}{(1+t^2)}}_{\in L^q} \cdot \underbrace{(1+t^2) \cdot |\varphi(t)|}_{\in L^\infty}\, dt.$$

□

15.3. Elementary operations on tempered distributions

15.3.1. Derivative of a tempered distribution

Definition 472.
Every tempered distribution T has derivatives (of any order) defined as follows:

1. $\langle T', \varphi \rangle = -\langle T, \varphi' \rangle, \quad \forall \varphi \in \mathscr{S};$
2. $\langle T'', \varphi \rangle = \langle T, \varphi'' \rangle, \quad \forall \varphi \in \mathscr{S};$
3. $\langle T''', \varphi \rangle = -\langle T, \varphi''' \rangle, \quad \forall \varphi \in \mathscr{S};$
4. $\ldots .$

Hence, for $k = 1, 2, 3, \ldots$,

$$\boxed{\langle T^{(k)}, \varphi \rangle = (-1)^k \langle T, \varphi^{(k)} \rangle, \quad \forall \varphi \in \mathscr{S}.}$$

Proposition 473.
The mapping

$$\mathscr{S}' \to \mathscr{S}', \quad T \mapsto T^{(k)}$$

is continuous, for all $k = 1, 2, 3, \ldots$.

Thus

$$T_n \to T \quad \text{in } S' \Longrightarrow T_n^{(k)} \to T^k \quad \text{in } S',$$

i.e. if

$$\lim_{n\to\infty} \langle T_n, \varphi \rangle = \langle T, \varphi \rangle, \qquad \forall \varphi \in \mathscr{S},$$

then

$$\lim_{n\to\infty} \langle T_n^{(k)}, \varphi \rangle = \langle T^{(k)}, \varphi \rangle, \qquad \forall \varphi \in \mathscr{S},$$

15.3.2. Multiplication by powers of t

Proposition 474.
The mapping

$$\mathscr{S}' \to \mathscr{S}', \quad T \mapsto t^k \cdot T$$

is continuous, for all $k = 1, 2, 3, \ldots$.

15.3.3. The dirac comb as a tempered distribution

Definition 475.
A sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ of complex numbers is slowly increasing if there exists an integer $N > 0$ and a constant A such that

$$|\alpha_n| \leq A \cdot |n|^N \qquad \text{for all sufficiently large } |n|.$$

Proposition 476.

Hyp — *Consider a sequence* $\{\alpha_n\}_{n\in\mathbb{Z}}$ *of complex numbers that is slowly increasing and a constant* $a > 0$

Concl — *Then*

$$T = \sum_{k\in\mathbb{Z}} \alpha_k \cdot \delta_{k\cdot a}$$

is a tempered distribution.

Proposition 477.
The Dirac comb Ш_a *is a tempered distribution.*

16

The Fourier transorm of tempered distributions

16.1. Definition and main properties

Definition 478.
We define the Fourier transform

$$\mathscr{F}_{\mathscr{S}'} : \mathscr{S}' \to \mathscr{S}', \quad T \mapsto \mathscr{F}_{\mathscr{S}'}[T] =: \hat{T}$$

through

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle, \qquad \forall \varphi \in \mathscr{S}.$$

Proposition 479.

Hyp *Consider a function $f \in L^2_{\mathbb{C}}(\mathbb{R})$.*
Hence f can be viewed as a tempered distribution T_f.
Moreover, its Fourier transform $\mathscr{F}_{L^2}[f(t)](\lambda) =: \hat{f}(\lambda)$, being an element in $L^2_{\mathbb{C}}(\mathbb{R})$, can be viewed as a tempered distirubution $T_{\hat{f}}$.

Concl *Then*

$$\widehat{T_f} = T_{\hat{f}},$$

i.e.

$$\langle \widehat{T_f}, \varphi \rangle = \int_{\mathbb{R}} \hat{f}(\lambda) \cdot \varphi(\lambda)\, d\lambda, \qquad \forall \varphi \in \mathscr{S}.$$

Proof. This follows from then facts, that

- $\mathscr{S} \subset L^2_{\mathbb{C}}(\mathbb{R})$;
- for all f and $g \in L^2_{\mathbb{C}}(\mathbb{R})$, on has

$$\int_{\mathbb{R}} \hat{f}(\lambda) \cdot g(\lambda)\, d\lambda = \int_{\mathbb{R}} f(t) \cdot \hat{g}(t)\, dt.$$

□

Proposition 480.

Hyp *Consider a function $f \in L^1_{\mathbb{C}}(\mathbb{R})$.*
Hence f can be viewed as a tempered distribution T_f.
Moreover, its Fourier transform $\mathscr{F}_{L^1}[f(t)](\lambda) =: \hat{f}(\lambda)$, being an element in $L^\infty_{\mathbb{C}}(\mathbb{R})$, can be viewed as a tempered distirubution $T_{\hat{f}}$.

Concl

$$\widehat{T_f} = T_{\hat{f}},$$

i.e.

$$\langle \widehat{T_f}, \varphi \rangle = \int_{\mathbb{R}} \hat{f}(\lambda) \cdot \varphi(\lambda) \, d\lambda, \qquad \forall \varphi \in \mathscr{S}.$$

Proposition 481.

Hyp *Let T be a tempered distribution.*

Concl *Then*

1. *For $k = 1, 2, 3, \ldots$, one has*

$$\begin{aligned} \hat{T}^{(k)} &= [(-2\pi i t)^k \cdot T]^{\wedge} \\ \widehat{T^{(k)}} &= (2\pi i \lambda)^k \hat{T}. \end{aligned}$$

2. *For $a \in \mathbb{R}$, one has*

$$\begin{aligned} \tau_a \hat{T} &= [e^{2\pi i a t} T]^{\wedge} \\ \widehat{\tau_a T} &= e^{-2\pi i a \lambda} \hat{T}. \end{aligned}$$

Proof for the first statement. We have, $\forall \varphi \in \mathscr{S}$,

$$\begin{aligned} \langle [(2\pi i t)^k T]^{\wedge}, \varphi \rangle &= \langle T, (2\pi i t)^k \cdot \hat{\varphi}(t) \rangle \\ &= \langle T, \widehat{\varphi^{(k)}} \rangle \\ &= (-1)^k \langle \hat{T}^{(k)}, \varphi \rangle \end{aligned}$$

and this gives the claim.

Remark that the proof of the other relations is similar! □

Proposition 482.

The Fourier transform

$$\mathscr{F}_{\mathscr{S}'}; \mathscr{S}' \to \mathscr{S}', \quad T \mapsto \mathscr{F}_{\mathscr{S}'}[T] = \hat{T}$$

is a linear, bi-continuous bijection, whose inverse is given by

$$\langle \mathscr{F}_{\mathscr{S}'}^{-1}[T], \varphi \rangle = \langle T, \mathscr{F}^{-1}[\varphi] \rangle, \qquad \forall \varphi \in \mathscr{S}.$$

Example 483.
One has, $\forall \varphi \in \mathscr{S}$,

$$\begin{aligned} \langle \hat{\delta}, \varphi \rangle &= \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) \\ &= \int_{\mathbb{R}} \varphi(t) \cdot e^{-2\pi i 0 t} \, dt = \int_{\mathbb{R}} 1 \cdot \varphi(t) \, dt \end{aligned}$$

Thus

$$\boxed{\hat{\delta} = 1 \qquad \text{in } \mathscr{S}'.}$$

Moreover

$$\begin{aligned} \langle \widehat{\delta_a}, \varphi \rangle &= \langle \delta_a, \hat{\varphi} \rangle = \hat{\varphi}(a) \\ &= \int_{\mathbb{R}} \varphi(t) \cdot e^{-2\pi i a t} \, dt. \end{aligned}$$

Thus

$$\boxed{\widehat{\delta_a}(\lambda) = e^{-2\pi i a \lambda} \qquad \text{in } \mathscr{S}'.}$$

Example 484.
Consider the tempered distribution T_f generated by

$$f(t) = e^{2\pi i a t}$$

(with a fixed $a \in \mathbb{R}$). Then

$$\begin{aligned} \langle \widehat{T_f}, \varphi \rangle &= \langle T_f, \hat{\varphi} \rangle \\ &= \int_{\mathbb{R}} f(t) \cdot \hat{\varphi}(\lambda) \, d\lambda = \int_{\mathbb{R}} e^{2\pi i a \lambda} \cdot \hat{\varphi}(\lambda) \, d\lambda \\ &= \mathscr{F}_{L^1}^{-1}[\hat{\varphi}(\lambda)](a) = \varphi(a), \end{aligned}$$

so that

$$\boxed{\widehat{T_{e^{2\pi i a t}}} = \delta_a.}$$

For $a = 0$, we get

$$\boxed{\widehat{T_1} = \delta.}$$

Recall that we have developed the formula

$$\boxed{Ш_a = \sum_{k\in\mathbb{Z}} \delta_{k\cdot a} = \frac{1}{a}\sum_{k\in\mathbb{Z}} e^{2\pi i \frac{k}{a} t}}$$

This relation will help us to compute the Fourier transformed of $Ш_a$.

Example 485.
Since the Fourier transform $\mathscr{F}_{\mathscr{S}'}$ is continuous, we have

$$\begin{aligned}\widehat{Ш_a}(\lambda) &= \sum_{k\in\mathbb{Z}} \widehat{\delta_{k\cdot a}}(\lambda) = \sum_{k\in\mathbb{Z}} e^{2\pi i k a \lambda} \\ &= \frac{1}{a}\cdot Ш_{1/a}(\lambda).\end{aligned}$$

Thus

$$\boxed{\widehat{Ш_a} = \frac{1}{a} Ш_{1/a}.}$$

16.2. The Fourier series viewed as Fourier transformed

Let us consider an a-periodic signal f with $f \in L^2_{\mathbb{C}}([0,a])$. Thus we have

$$\lim_{n\to\infty} \|f - f_n\|_{L^2} = 0$$

if we set

$$f_n(t) = \sum_{k=-n}^{n} c_k e^{2\pi i \frac{k}{a} t}$$

with

$$c_k = \frac{1}{a}\int_0^a f(t)\cdot e^{-2\pi i \frac{k}{a} t}\, dt.$$

Remark that, due to the L^2 convergence, we have

$$T_{f_n} \to T_f \quad \text{in } \mathscr{S}'.$$

Thus we get, $\forall \varphi \in \mathscr{S}$,

$$\begin{aligned}
\langle \widehat{T_f}, \varphi \rangle &= \langle T_f, \hat{\varphi} \rangle = \lim_{n\to\infty} \langle T_{f_n}, \hat{\varphi} \rangle \\
&= \lim_{n\to\infty} \sum_{k=-n}^{n} c_k \langle e^{2\pi i \frac{k}{a}\lambda}, \hat{\varphi}(\lambda) \rangle \\
&= \lim_{n\to\infty} \sum_{k=-n}^{n} c_k \varphi\left(\frac{k}{a}\right) \\
&= \lim_{n\to\infty} \left\langle \sum_{k=-n}^{n} c_k \cdot \delta_{k/a}, \varphi \right\rangle \\
&= \left\langle \sum_{k\in\mathbb{Z}} c_k \cdot \delta_{k/a}, \varphi \right\rangle .
\end{aligned}$$

Proposition 486.
Let us consider an a-periodic signal f with $f \in L^2_{\mathbb{C}}([0, a])$. Thus we have

$$\lim_{n\to\infty} \|f - f_n\|_{L^2} = 0$$

if we set

$$f_n(t) = \sum_{k=-n}^{n} c_k e^{2\pi i \frac{k}{a} t}$$

with

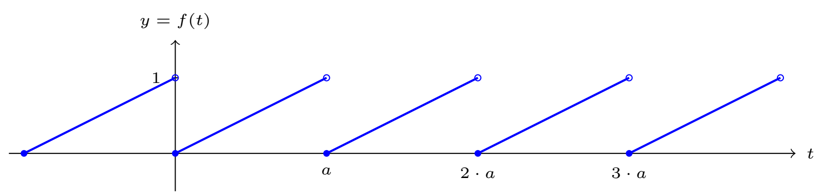
$$c_k = \frac{1}{a} \int_0^a f(t) \cdot e^{-2\pi i \frac{k}{a} t} \, dt.$$

Then

$$\boxed{\widehat{T_f} = \sum_{k\in\mathbb{Z}} c_k \cdot \delta_{k/a}.}$$

Example 487.
Consider again the a-periodic signal defined by

$$f(t) = t/a, \qquad \text{for } t \in [0, a[$$



An easy computation shows that

$$c_0 = \frac{1}{2}$$

and

$$c_k = \frac{i}{2\pi} \cdot \frac{1}{k}, \qquad \forall k \in \mathbb{Z} \setminus \{0\}.$$

Thus the modulus of the Fourier transformed of f is represented in the following way: